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ON RANDOM SUBSETS OF PROJECTIVE SPACES

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Let us define a *random* $PG(r-1,q)$-*process* $\{\omega_r(M)\}_{M=0}^{(q^r-1)/(q-1)}$ as a Markov chain of subsets of elements of projective space $PG(r-1,q)$, which starts with the empty set, and, for $M=1,2,\ldots,(q^r-1)/(q-1)$, $\omega_r(M)$ is obtained by adding to $\omega_r(M-1)$ a new randomly chosen element of $PG(r-1,q)$. Clearly, one may also view a random submatroid $\omega_r(M)$ as a subset chosen at random from all M-element subsets of the projective geometry $PG(r-1,q)$. We say that a subset S of $PG(r-1,q)$ is *independent* if it spans in it a subspace of dimension $|S|-1$. By the *rank* $\varrho(T)$ of a subset $T\subseteq PG(r-1,q)$ we mean the size of the largest independent set contained in T. In this note we show that for large r typically the rank $\varrho(\omega_r(M))$ does not differ from $|M|$ very much.

The analogous problem for $\omega_r(p)$—a random set in which each element of $PG(r-1,q)$ appears independently with probability p—was considered by Kelly and Oxley in [2] (see also Kordecki [3]). They proved that if $k(r)$, $0\le k(r)\le r$, is a function of r for which $\liminf_{r\to\infty} k(r)/r>0$ and $p'(r)/rq^{-r}\to\infty$ then a.s. $r(\omega_r(p'(r))\ge k(r)$, whereas for $p''(r)/rq^{-r}\to 0$ a.s. we have $r(\omega_r(p''(r))\le k(r)$. (Here and below a.s. means "with probability tending to 1 as $r\to\infty$".) We shall give a simple argument which shows that a much stronger result holds.

THEOREM. *If* $r-M(r)\to\infty$ *as* $r\to\infty$ *then a.s.* $\varrho(\omega_r(M))=M$.

P r o o f. To simplify computations let us introduce $\{\widehat\omega_r(M)\}_{M=0}^{\infty}$ as a nondecreasing sequence of subsets of $PG(r-1,q)$ which starts with the empty set and at each step we add to $\widehat\omega_r(M)$ a randomly chosen element of $PG(r-1,q)$. Although in this case it may happen that $\widehat\omega_r(M)=\widehat\omega_r(M+1)$, clearly $\widehat\omega_r(M)$ might be identified with $\omega_r(M)$ whenever $|\widehat\omega_r(M)|=M$. Recall that for every $k=1,2,\ldots,r$ each subspace of $PG(r-1,q)$ of rank k contains

$$[k]=\frac{q^k-1}{q-1}$$

elements, in particular, $PG(r-1,q)$ consists of $(q^r-1)/(q-1)$ points. Hence the probability that $|\widehat{\omega}_r(2r)| < 2r$ is less than

$$r^2(q-1)/(q^r-1) \to 0\ .$$

Thus, we have shown the following fact.

FACT 1. *A.s.* $|\widehat{\omega}_r(i)| = i$ *for every* $i \le 2r$. ∎

Hence, the asymptotic properties of the first $2r$ stages of the random PG$(r-1,q)$-process $\{\omega_r(M)\}_{M=0}^{(q^r-1)/(q-1)}$ are identical with those of $\{\widehat{\omega}_r(M)\}_{M=0}^{\infty}$.

Let $1 \le M \le r$. The probability that $\varrho(\widehat{\omega}_r(M)) = M$, i.e. that each new point is picked outside the subspace generated by the already chosen points is given by

$$\prod_{k=1}^{M}\left(1-\frac{[k]}{[r]}\right) = \prod_{k=1}^{M}\left(1-\frac{q^k-1}{q^r-1}\right) = \prod_{k=1}^{M}\left(1-q^{k-r}+O(q^{-r})\right)$$
$$= (1+O(Mq^{-r}))\prod_{k=1}^{M}\left(1-q^{k-r}\right)\ .$$

Moreover, if we assume that $r - M \to \infty$ then

$$\prod_{k=1}^{M}\left(1-q^{k-r}\right) = \exp\Big(-\sum_{k=1}^{M}(q^{k-r}+O(q^{2k-2r}))\Big)$$
$$= \exp\left(-q^{-r}\frac{q^{M+1}-1}{q-1}+O(q^{2M+2-2r})\right) \to 1\,.$$

Hence a.s. $\varrho(\widehat{\omega}_r(M)) = M$, and due to Fact 1, a.s. $\varrho(\omega_r(M)) = M$. ∎

Now, let us look at the value of $\varrho(\omega_r(M))$ when M approaches r. More precisely, let M_{cr} denote the minimal value of M for which $\varrho(\omega_r(M)) = r$ and set $u_r = r - M_{\mathrm{cr}}$. Again, instead of studying u_r we shall consider the corresponding random variable $\widehat{u}_r$ defined for $\{\widehat{\omega}_r(M)\}_{M=0}^{\infty}$.

To find the distribution of $\widehat{u}_r$ it is enough to notice that $\widehat{u}_r$ is the sum of the random variables $\widehat{u}_r^{(k)}$ which count the number of points picked in the subspace generated by the already chosen points when the rank of this subspace equals k. Each $\widehat{u}_r^{(k)}$ has a geometric distribution, thus, for example, for the expectation of $\widehat{u}_r$ we have

$$\mathrm{E}\widehat{u}_r = \sum_{k=1}^{r-1}\widehat{u}_r^{(k)} = \sum_{k=1}^{r-1}\frac{(q^k-1)/(q^r-1)}{1-(q^k-1)/(q^r-1)} = (1+o(1))\sum_{i=1}^{\infty}\frac{q^{-i}}{1-q^{-i}}\ .$$

From the above result and Markov's inequality it follows immediately that $\widehat{u}_r$ (and, due to Fact 1, also u_r) is bounded in probability.

FACT 2. *Let $\gamma(r) \to \infty$. Then a.s. both $\widehat{u}_r$ and u_r are less than $\gamma(r)$.* ■

Since the generating function of $\widehat{u}_r^{(k)}$ equals $(1-q^{-k})/(1-sq^{-k})$, the generating function of $\widehat{u}_r$ is given by

$$g(s) = \prod_{k=1}^{r-1} \frac{1-q^{-k}}{1-sq^{-k}} = (1+O(sq^{-r}))\,\beta \prod_{k=1}^{\infty}(1-sq^{-k})^{-1}$$

where we set $\beta = \prod_{k=1}^{\infty}(1-q^{-k})$.

The well known Euler formula (see, for example, [1], p. 19, Corollary 2.2) says that

$$\prod_{k=0}^{\infty}(1-st^k)^{-1} = 1 + \sum_{k=1}^{\infty} \frac{s^k}{(1-t)(1-t^2)\dots(1-t^k)}$$

for $|s|<1$ and $|t|<1$, so, for $g(s)$ we get immediately

$$g(s) = \beta(1+O(q^{-r}))\left[1 + \sum_{k=1}^{\infty} \frac{s^k q^{-k}}{\prod_{i=1}^{k}(1-q^{-i})}\right].$$

Thus we arrive at the following formula for the limit distributions of $\widehat{u}_r$ and u_r.

FACT 3.

$$\lim_{r\to\infty} \operatorname{Prob}\{u_r = k\} = \lim_{r\to\infty} \operatorname{Prob}\{\widehat{u}_r = k\} = \begin{cases} \beta & \text{if } k=0, \\ \beta q^{-k} \big/ \prod_{i=1}^{k}(1-q^{-i}) & \text{if } k \ge 1. \end{cases}$$ ■

Clearly, our results (and model) are much more precise than those used by Kelly and Oxley in [2]. For instance, the limit value of the probability that $\varrho(\omega_r(p)) = r$ follows easily from the Theorem, Fact 2 and the fact that the number of points which belong to $\omega_r(p)$ is binomially distributed.

COROLLARY. *Let a be a real number and $p(r) = (r+a\sqrt{r})(q-1)/(q^r-1)$. Then*

$$\lim_{r\to\infty} \operatorname{Prob}\{\varrho(\omega_r(p)) = r\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2}\,dx\,.$$ ■

(The above form of the threshold function of $p(r)$ was anticipated by Kordecki in [3], although the limit probability of the event $\varrho(\omega_r(p)) = r$ conjectured in [3] turns out to be incorrect.)

Finally, we should point out that the only property of projective space we have used in our argument is the fact that the subspaces of $PG(r-1,q)$ form a lattice with the Jordan–Dedekind property in which for each element e of rank k there exist roughly q^{k-1} atoms a such that $a \preceq e$.

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