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INVARIANT MEASURES OF THE PAIR: STATE, APPROXIMATE FILTERING PROCESS

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1. Introduction. First we formulate our problem in a more general set-up. Let $X = (\Omega_1 = E^N, F^1 = \mathcal{E}^N, x_t, P_x^1)$ be a Markov process with transition operator $P(x, dz)$ and values in a measurable space $(E, \mathcal{E})$. Let (w_i), $i = 1, 2, \ldots$, be a sequence of i.i.d. $\mathbb{R}^d$-valued random vectors defined on a probability space $(\Omega_2 = \mathbb{R}^d,\ F^2 = \mathcal{B}(\mathbb{R}^d), P^2)$. Assume that the w_i have positive density $g(y)$ with respect to d-dimensional Lebesgue measure. Define the probability space $\Omega = \Omega_1 \times \Omega_2$, $F = F^1 \otimes F^2$, $P_x = P_x^1 \otimes P^2$. Clearly X defined on Ω by $x_t(\omega_1, \omega_2) = x_t(\omega_1)$ for $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$ is independent of (w_i) where $w_i(\omega_1, \omega_2) = w_i(\omega_2)$, and has the same transition function $P(x, dz)$. Assume that we cannot observe (x_t) directly. The only information about X is given through the $\mathbb{R}^d$-valued observation process $Y = (y_n)_{n=0,1,\ldots}$, $y_0 = 0$, $y_n = h(x_n, w_n)$, where for x fixed, $h(x, \cdot)$ is a diffeomorphism of $\mathbb{R}^d$, i.e., a 1-1, C^1 transformation with nonvanishing Jacobian.

For a given initial law ν of X, define the so-called *filtering process*

$$\pi_n^{(\nu)}(f) \stackrel{\text{def}}{=} E_\nu\{f(x_n) | y_1, \ldots, y_n\} \qquad P_\nu \text{ a.s.}$$

for $f \in B(E)$, the space of bounded measurable functions on E. From the Kallianpur–Striebel formula (see (6) of [6]) we obtain

$$\pi_{n+1}^{(\nu)}(f) = \frac{\sigma_{n+1}^{(\nu)}(f)}{\sigma_{n+1}^{(\nu)}(1)} \stackrel{\text{def}}{=} S(f, y_{n+1}, \pi_n^{(\nu)}) \tag{1}$$

with

$$\begin{aligned} \sigma_{n+1}^{(\nu)}(f) &= \int_E \int_E f(z) g(w(z, y_{n+1})) |\text{Jacobian } w(z, y_{n+1})| P(x, dz)\, \sigma_n^{(\nu)}(dx)\,, \\ \sigma_0^{(\nu)}(\cdot) &= \nu(\cdot)\,, \end{aligned} \tag{2}$$

where $w(x, z)$ for x fixed is the inverse function to $h(x, z)$.

The filtering process $\pi_n^{(\nu)}$ considered as a process on $\mathcal{P}(E)$, the space of probability measures on E, is Markov (Lemma 3 of [6]) with respect to the observation σ-field $G_n \stackrel{\text{def}}{=} \sigma(y_1, \ldots, y_n)$. Therefore one can study the limit behaviour of $\pi_n^{(\nu)}$ as $n \to \infty$. This problem was the subject of the papers [2], [3], [6]. For further reference let us recall

PROPOSITION (Theorem 2 in [6]). *Assume that X has a unique invariant measure μ. Then there exists a unique invariant measure for the filtering process $\pi_n^{(\nu)}$ if and only if*

$$\limsup_{n\to\infty} \int_E |P^n f(x) - \mu(f)|\, \mu(dx) = 0 \tag{3}$$

for each $f \in C(E)$, the space of bounded continuous functions on E.

Unfortunately, in practical applications we usually do not know the initial law of X. Therefore we construct recursively an *approximate filtering process* $\pi_n^{(\nu),\eta}$. Namely, we replace in (1) the actual initial law ν by $\eta \in \mathcal{P}(E)$, i.e., we set

$$\pi_0^{(\nu),\eta}(\cdot) = \eta(\cdot), \qquad \pi_{n+1}^{(\nu),\eta}(\cdot) = S(\cdot, y_{n+1}, \pi_n^{(\nu),\eta}). \tag{4}$$

The process $\pi_0^{(\nu),\eta}$ is not Markov unless $\nu = \eta$. To regain the Markov property we consider the pair: state + approximate filtering process.

LEMMA. *The pair $(x_n, \pi_n^{(\nu),\eta})$ forms a homogeneous Markov process.*

Proof. It suffices to observe the following: for any bounded measurable function f on $E \times \mathcal{P}(E)$ we have

$$\begin{aligned} &E\{f(x_{n+1}, \pi_{n+1}^{(\nu),\eta}) | x_n, \pi_n^{(\nu),\eta}\} \\ &\quad = E\{E\{f(x_{n+1}, S(\cdot, y_{n+1}, \pi_{n+1}^{(\nu),\eta})) | x_n, x_{n+1}, \pi_n^{(\nu),\eta}\} | x_n, \pi_n^{(\nu),\eta}\} \\ &\quad = \int_E \int_{\mathbb{R}^d} f(u, S(\cdot, h(u,z), \pi_n^{(\nu),\eta})) g(z)\, dz\, P(x_n, du). \end{aligned}$$

The problem of limit behaviour of the state + approximate filtering process $(x_n, \pi_n^{(\nu),\eta})$ was posed in the continuous time case by Professor H. Kushner in [3]. More precisely, he conjectures that if X has a unique invariant measure μ and (3) is satisfied, then also for any $\eta \in \mathcal{P}(E)$ the process $(x_n, \pi_n^{(\nu),\eta})$ has a unique invariant measure.

Below, we are only able to solve the discrete time version of the problem, assuming additionally that X is a finite state ergodic Markov process and that the observation process is 1-dimensional of the form

$$y_n = h(x_n) + w_n \tag{5}$$

where h is injective and (w_n) forms a sequence of i.i.d. $N(0,1)$ random variables.

2. Main theorem. From now on $E = \{a_1, \ldots, a_m\}$ and X is a Markov process with transition probability matrix $p_{ij} = P\{x_1 = a_j | x_0 = a_i\}$, $i, j = 1, \ldots, m$. Moreover, the observation process Y is given by (5).

THEOREM. *Assume that X is aperiodic and there are no transient states, i.e., for each $i, j = 1, \ldots, m$ there exists k such that $p_{ij}^k = P\{x_k = a_j \mid x_0 = a_i\} > 0$. Then for any $\eta \in \mathcal{P}(E)$ the process $(x_n, \pi_n^{(\nu),\eta})$ has a unique invariant measure.*

Let us make some comments on the result just formulated. Clearly, for a Markov process X satisfying the requirements of the Theorem condition (3) holds. Thus by virtue of the Proposition there exists a unique invariant measure for the filtering process $(\pi_n^{(\nu)})$. Nevertheless, at least at first glance, it is not clear how to recover from this fact the uniqueness of invariant measure for the state + approximate filtering process. The principal significance of the Theorem lies in that it allows us to apply limit theorems, and in this way, for example, to identify Cesàro mean square errors of the approximate filtration (for details see Section 8 of [3]).

3. Proof of Theorem. In our case, for $y \in \mathbb{R}$, $\mu \in \mathcal{P}(E)$, $\mu(a_i) = \mu_i$, we have

$$S(a_j, y, \mu) = \frac{\sigma(a_j, y, \mu)}{\sum_{i=1}^m \sigma(a_i, y, \mu)} \tag{6}$$

with

$$\sigma(a_j, y, \mu) = \exp[yh(a_j) - \tfrac{1}{2}(h(a_j))^2] \sum_{k=1}^m p_{kj}\mu_k \,. \tag{7}$$

To simplify notation we consider the measures $S(\cdot, y, \mu)$ and $\mu(\cdot)$ as m-dimensional vectors with ith coordinates equal to $S(a_i, y, \mu)$, $\mu(a_i)$ respectively. Then

$$\sigma(y, \mu) = e(y) P^T \mu$$

with

$$e(y) = \mathrm{diag}[\exp(yh(a_i) - \tfrac{1}{2}(h(a_i))^2)] \quad \text{and} \quad P = (p_{ij})_{i,j=1,\ldots,m} \,.$$

We first prove that for the approximate filtering process there exists an invariant set with nonempty interior.

For simplicity, we start with the case where $p_{ij} > 0$ for each $i, j = 1, \ldots, m$. Let $\eta \in \mathcal{P}(E)$ and

$$f^\eta : \mathbb{R}^m \ni (s_1, \ldots, s_m) \mapsto e(s_1)P^T e(s_2)P^T \ldots e(s_m)P^T \eta \,. \tag{8}$$

Then

$$\text{(9)}\qquad \frac{\partial f^\eta}{\partial s_j}(s_1,\dots,s_m) = e(s_1)P^T e(s_2)P^T \dots e(s_{j-1})P^T B e(s_j)P^T \dots e(s_m)P^T\eta$$

with $B = \operatorname{diag}[b(a_i)]$.

By the Brouwer fixed point theorem (Thm. II.7.3 of [1]), for each $y \in \mathbb{R}$ there exists $\eta(y) \in \mathcal{P}(E)$ such that $S(y,\eta(y)) = \eta(y)$. Thus the Jacobian of f^η with $\eta(y)$ substituted for η at $y \in \mathbb{R}$ is equal to

$$\text{(10)}\qquad c|B\eta(y)e(y)P^T B\eta(y)\dots(e(y)P^T)^{m-1}B\eta(y)| \quad \text{with } c > 0\,.$$

Assume $h(a_1) > h(a_2) > \dots > h(a_m)$, $h(a_1) > 0$. A direct calculation of (10) gives

$$\begin{aligned}\text{(11)}\quad & \text{Jacobian}\, f^{\eta(y)}(y,\dots,y)\\ &= k\exp\{[(m-1)h(a_1) + (m-2)h(a_1) + h(a_2)\\ &\quad + (m-3)h(a_1) + h(a_3) + \dots + h(a_1) + h(a_{m-1}) + h(a_m)]y\}\\ &\quad + \text{ terms with smaller powers of } e^y\,.\end{aligned}$$

Therefore for y sufficiently large, Jacobian $f^{\eta(y)}(y,\dots,y) \ne 0$.

In the case $h(a_1) = 0$, clearly $h(a_m) < 0$ if $m > 1$, and then a representation similar to (11) shows that for y sufficiently small, Jacobian $f^{\eta(y)}(y,\dots,y) \ne 0$.

In both cases from the regular mapping theorem (5.5.1 of [4]) it follows that the invariant set for the approximate filtering process starting with the measure $\eta(y)$ has nonempty interior.

If $m = 1$ there is nothing to prove, since the initial law is known.

If X is aperiodic and all states are communicative then, since $p_{ij}^n \to \mu_j > 0$ as $n \to \infty$, there exists n such that $p_{ij}^n > 0$, for $i,j = 1,\dots,m$. Fix $\overline{y} \in \mathbb{R}$ and define $G = P^T e(\overline{y})P^T \dots e(\overline{y})P^T$, where $e(\overline{y})$ is repeated $n-1$ times.

Let now

$$f^\eta : \mathbb{R}^m \ni (s_1,\dots,s_m) \mapsto e(s_1)Ge(s_2)\dots e(s_m)G\eta\,.$$

By the Brouwer theorem, for each $y \in \mathbb{R}$ there exist $\eta(y) \in \mathcal{P}(E)$ and a positive constant $k(y)$ such that

$$e(y)G\eta(y) = k(y)\eta(y)\,.$$

Then by the same argument as in the case $p_{ij} > 0$, $i,j,\dots,m$,

$$\begin{aligned}&\text{Jacobian} f^{\eta(y)}(y,\dots,y)\\ &\qquad = c|B\eta(y)e(y)GB\eta(y)\dots(e(y)G)^{m-1}B\eta(y)| \ne 0\end{aligned}$$

and for the approximate filtering process there is an invariant set with nonempty interior. But the filtering process and the approximate filtering process have the same invariant sets. Thus the filtering process possesses an invariant set with nonempty interior. Therefore, because of the Proposition there is only one invariant set. In fact, assume that A is invariant and take an open set $\mathcal{O} \subset A$. Then for $f \in C(\mathcal{P}(E))$, $0 \neq \operatorname{supp} f \subset \mathcal{O}$, $f \geq 0$, by Theorem 3 of [6]

$$E_\nu\{f(\pi_n^{(\nu)})\} \to \Phi(f) > 0, \quad \text{for any } \nu \in \mathcal{P}(E), \text{ as } n \to \infty,$$

where Φ is an invariant measure for the filtering process, and if there is an invariant set $\widetilde{A}$ disjoint from A, then for $\eta \in \widetilde{A}$, $E_\eta\{f(\pi_n^{(\eta)})\} \to 0$, a contradiction.

In summary, the approximate filtering process has a unique invariant set. Thus by virtue of Theorem I.21 of [5] we obtain the uniqueness of invariant measure for the state + approximate filtering process. This completes the proof of the Theorem.

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