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SOME INDEFINITE METRICS AND COVARIANT DERIVATIVES OF THEIR CURVATURE TENSORS

BY

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1. Introduction. Let (M,g) be a Riemannian or pseudo-Riemannian manifold.

We denote its curvature tensor, Ricci tensor, scalar curvature and Weyl conformal curvature tensor by R, S, K and C respectively, while ∇ stands for covariant differentiation with respect to g.

Nomizu and Ozeki proved the following remarkable result [11]:

THEOREM A. *In a Riemannian manifold, if* $\nabla^t R = 0$ *for some* $t \geq 1$, *then* $\nabla R = 0$.

Tanno extended this theorem as follows:

THEOREM B (see [17], Theorem 2). *Let* (M,g) *be a Riemannian manifold.*

(a) *If* $\nabla^t S = 0$ *for some* $t \geq 1$, *then* $\nabla S = 0$.
(b) *If* $\nabla^t C = 0$ *for some* $t \geq 1$, *then* $\nabla C = 0$.
(c) *If* $\nabla^t K = 0$ *for some* $t \geq 1$, *then* $K =$ *constant.*
(d) *If* $\nabla^t P = 0$ *for some* $t \geq 1$, *then* $\nabla P = 0$ *and* $\nabla R = 0$, *where* P *denotes the Weyl projective curvature tensor of* (M,g).

Moreover, investigating Riemannian manifolds with conformally related metrics, Nickerson proved

THEOREM C (see [10], Theorem 4.1). *A conformally recurrent manifold with* $C \neq 0$ *cannot be conformal to a Riemannian locally symmetric one.*

In connection with the above theorems, an interesting question arises whether these results are valid for pseudo-Riemannian manifolds.

Unfortunately, for a 4-dimensional indefinite metric Kaĭgorodov has proved [8] that Theorem A fails in general.

The present paper deals with examples (Examples 1 and 2) of certain n-dimensional ($n \geq 4$) metrics which show that neither Theorems A, B (except case (c), which will be treated in a subsequent paper) nor Nickerson's Theorem C remain true for indefinite metrics.

We shall also prove (Corollary 7) the existence of non-recurrent Ricci-recurrent simple conformally recurrent metrics which are not conformal to any essentially conformally symmetric one.

Throughout this paper, all manifolds under consideration are assumed to be connected and of class C^∞.

The present author is grateful to Professors K. Nomizu and U. Simon whose interesting questions concerning Theorem A called the author's attention to problems considered in this paper.

2. Preliminaries. In the sequel we shall need the following definitions and lemmas:

An n-dimensional ($n \geq 4$) Riemannian or pseudo-Riemannian manifold is called *conformally symmetric* [2] if its Weyl conformal curvature tensor

$$C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{ij}S_{hk} - g_{ik}S_{hj} + g_{hk}S_{ij} - g_{hj}S_{ik}) + \frac{K}{(n-1)(n-2)}(g_{ij}g_{hk} - g_{hj}g_{ik}) \tag{1}$$

is parallel, i.e. if $\nabla C = 0$.

Clearly, the class of conformally symmetric manifolds contains all locally symmetric ones as well as all conformally flat manifolds of dimension $n \geq 4$.

The existence of essentially conformally symmetric manifolds, i.e. conformally symmetric manifolds which are neither conformally flat nor locally symmetric, will be shown in Section 3 (see also [3] and [4]). Such manifolds cannot have definite metrics [5].

Let M be a manifold with a (possibly indefinite) metric g. A smooth tensor field T on M will be called *recurrent* if

$$T_{i_1\ldots i_q}T_{j_1\ldots j_q,l} = T_{i_1\ldots i_q,l}T_{j_1\ldots j_q}\,, \tag{2}$$

where the comma denotes (here and in the sequel) covariant differentiation with respect to g.

Every parallel tensor field is therefore recurrent.

Condition (2) states that at each point $x \in M$ such that $T(x) \neq 0$ there exists a (unique) covariant vector a (called the *recurrence vector* of T) which satisfies

$$T_{i_1\ldots i_q,l} = a_l T_{i_1\ldots i_q}\,. \tag{3}$$

The above definition of recurrency differs slightly from the classical one, i.e. that given by (3). Obviously, both definitions are equivalent on the subset of M where T does not vanish.

A Riemannian or pseudo-Riemannian manifold (M, g) will be called *recurrent* [18] (*Ricci-recurrent* [12]) if its curvature tensor (Ricci tensor) is recurrent.

Following Adati and Miyazawa [1], an n-dimensional ($n \geq 4$) manifold with a possibly indefinite metric will be called *conformally recurrent* if its Weyl conformal curvature tensor is recurrent.

Clearly, the class of conformally recurrent manifolds contains all conformally symmetric ones as well as all recurrent manifolds of dimension $n \geq 4$.

A conformally recurrent manifold (M, g) is said to be *simple* if its metric is locally conformal to a non-conformally flat conformally symmetric one, i.e., if for each point $x \in M$ there exist a neighbourhood U of x and a function p on U such that $\overline{g} = (\exp 2p)g$ is a non-conformally flat conformally symmetric metric.

Obviously, every non-conformally flat conformally symmetric manifold is necessarily simple conformally recurrent.

Simple conformally recurrent manifolds can be characterized as follows:

LEMMA 1 (see [14], Theorem 1). *A conformally recurrent manifold is simple conformally recurrent if and only if* (i) $C \neq 0$ *everywhere* (*which*, *in view of* (2), *implies*

$$C_{hijk,l} = a_l C_{hijk} \tag{4}$$

for some vector field a_j, *the recurrence vector of* C), (ii) *the recurrence vector is locally a gradient* ($a_{i,j} = a_{j,i}$), *and* (iii) *the Ricci tensor* S *is a Codazzi tensor* ($S_{ij,l} = S_{il,j}$).

The existence of non-simple conformally recurrent metrics with $C \neq 0$ has been established in [15].

In the general case, we have

LEMMA 2 (see [13], Theorem 1). *Let* (M, g) *be conformally recurrent. If* M *admits a function* p *such that* $(M, \overline{g})$ *with* $\overline{g} = (\exp 2p)g$ *is conformally recurrent*, *then*

$$p_l C^h{}_{ijk} + p_j C^h{}_{ikl} + p_k C^h{}_{ilj} = 0 \tag{e}$$

everywhere on M, $p_j = \partial_j p$.

(h) *At each point* $x \in M$ *such that* $C(x) \neq 0$ *we have* $\overline{a}_j = a_j - 4p_j$ *and* $p^r p_r = 0$, $\overline{a}_j$ *and* a_j *being the recurrence vectors of* $\overline{C}$ *and* C *respectively.*

LEMMA 3 (see [13], Theorem 2). *Let* (M, g) *be conformally recurrent. If* p *is a function on* M *satisfying condition* (e), *then* $(M, \overline{g})$ *with* $\overline{g} = (\exp 2p)g$ *is conformally recurrent.*

LEMMA 4 (see [13], Theorem 3). *Let* (M, g) *and* $(M, \overline{g})$ *be conformally symmetric. If* $\overline{g} = (\exp 2p)g$ *and* p *is a non-constant function on* M, *then both* (M, g) *and* $(M, \overline{g})$ *are conformally flat.*

The following lemma is a generalization of a result of Matsumoto [9]:

LEMMA 5. *Let (M,g) be a Riemannian or pseudo-Riemannian manifold with* $\dim M \geq 3$. *Then for each* $t \geq 1$,

$$\nabla^t P = 0 \quad \textit{if and only if} \quad \nabla^t R = 0.$$

Proof. Suppose that $\nabla^t P = 0$. Then, by the definition of P, we have

$$R_{hijl,q_1\ldots q_t} = \frac{1}{n-1}(g_{hl}S_{ij,q_1\ldots q_t} - g_{hj}S_{il,q_1\ldots q_t}),$$

whence,

$$S_{hl,q_1\ldots q_t} = \frac{1}{n}K_{,q_1\ldots q_t}g_{hl}.$$

But the last equation, in view of $S^r{}_{j,r} = \frac{1}{2}K_{,j}$, implies $\frac{1}{2}K_{,lq_2\ldots q_t} = \frac{1}{n}K_{,lq_2\ldots q_t}$. Hence, $S_{hl,q_1\ldots q_t} = 0$ and, consequently, $\nabla^t R = 0$. The converse implication is trivial. This completes the proof.

Remark 1. Lemma 5 seems to belong to the folklore. We have included its proof for completeness.

LEMMA 6. *Let* $\overline{g}_{ij} = (\exp 2p)g_{ij}$. *Then we have* ([7], pp. 89–90):

$$\overline{\left\{ {h \atop i\ j} \right\}} = \left\{ {h \atop i\ j} \right\} + \delta^h_i p_j + \delta^h_j p_i - p^h g_{ij}, \tag{5}$$

$$\overline{S}_{ij} = S_{ij} + (n-2)(p_{i,j} - p_i p_j) + (p^r{}_{,r} + (n-2)p^r p_r)g_{ij}, \tag{6}$$

$$\overline{C}^h{}_{ijk} = C^h{}_{ijk}, \tag{7}$$

where $p^h = g^{hr}p_r$.

3. Basic examples. The following definitions will be convenient:

Let (M,g) be a pseudo-Riemannian manifold. If its curvature (Ricci) tensor satisfies $\nabla^t R = 0$ ($\nabla^t S = 0$) for some $t \geq 2$ and $\nabla^{t-1}R$ ($\nabla^{t-1}S$) does not vanish everywhere, then (M,g) is called *t-symmetric* (*Ricci t-symmetric*). Similarly, if for the Weyl conformal (projective) curvature tensor the condition $\nabla^t C = 0$ ($\nabla^t P = 0$) holds for some $t \geq 2$ and $\nabla^{t-1}C$ ($\nabla^{t-1}P$) does not vanish everywhere, then (M,g) is said to be *conformally* (*projectively*) *t-symmetric*.

In this section each Latin index runs over $1,2,\ldots,n$, and each Greek index over $2,3,\ldots,n-1$. Moreover, the comma (as well as ∇) denotes covariant differentiation with respect to g.

EXAMPLE 1. *Let M denote the Euclidean n-space* ($n \geq 4$) *endowed with the indefinite metric g_{ij} given by*

$$g_{ij}dx^i dx^j = Q(dx^1)^2 + k_{\lambda\mu}dx^\lambda dx^\mu + 2dx^1 dx^n, \tag{8}$$

$$Q = (Ak_{\lambda\mu} + c_{\lambda\mu})x^\lambda x^\mu, \tag{9}$$

where $[k_{\lambda\mu}]$ *is an arbitrary symmetric non-singular constant matrix,* $[c_{\lambda\mu}]$ *is an arbitrary symmetric non-zero constant matrix satisfying* $k^{\alpha\beta}c_{\alpha\beta} = 0$ *with* $[k^{\lambda\mu}] = [k_{\lambda\mu}]^{-1}$, *and* A *is an arbitrary smooth non-constant function of* x^1 *only. Then*:

(i) M *is essentially conformally symmetric.*

(ii) M *is Ricci-recurrent and its scalar curvature vanishes everywhere.*

(iii) M *is not recurrent, but for each* $x \in M$ *such that* $(\nabla R)(x) \neq 0$ *there exists a vector* b *which satisfies* $R_{hijk,lm} = b_m R_{hijk,l}$. *The last condition states that* ∇R *is recurrent.*

(iv) *If*

$$A = \sum_{l=0}^{t-1} q_l (x^1)^l \,, \tag{10}$$

where $t \geq 2$, $q_i =$ const. $(i = 0, 1, \ldots, t-1)$ *and* $q_{t-1} \neq 0$, *then* M *is* t*-symmetric and Ricci* t*-symmetric.*

Proof. One can easily check that in the metric (8) the only Christoffel symbols not identically zero are

$$\left\{ {\lambda \atop 1\ 1} \right\} = -\frac{1}{2} k^{\lambda\omega} Q_{.\omega} \,, \quad \left\{ {n \atop 1\ 1} \right\} = \frac{1}{2} Q_{.1} \,, \quad \left\{ {n \atop 1\ \lambda} \right\} = \frac{1}{2} Q_{.\lambda} \,, \tag{11}$$

where the dot denotes partial differentiation with respect to coordinates.

Moreover, in view of the formula

$$\begin{aligned} R_{hijk} = {} & \frac{1}{2}(g_{hk.ij} + g_{ij.hk} - g_{hj.ik} - g_{ik.hj}) \\ & + g_{pq} \left(\left\{ {p \atop h\ k} \right\} \left\{ {q \atop i\ j} \right\} - \left\{ {p \atop h\ j} \right\} \left\{ {q \atop i\ k} \right\} \right) \end{aligned}$$

it follows that the only components R_{hijk} not identically zero are ([16], p. 179)

$$R_{1\lambda\mu 1} = \tfrac{1}{2} Q_{.\lambda\mu} \,. \tag{12}$$

It can also be found that

$$S_{11} = \tfrac{1}{2} k^{\alpha\beta} Q_{.\alpha\beta} \tag{13}$$

and that all other components of S are identically zero.

By an elementary computation, we can easily show that the only components of C, ∇S, ∇R and ∇C not identically zero are [14]

$$\begin{aligned} & C_{1\lambda\mu 1} = \frac{1}{2}\left(Q_{.\lambda\mu} - \frac{1}{n-2} k_{\lambda\mu} (k^{\alpha\beta} Q_{.\alpha\beta}) \right), \qquad S_{11,j} = \frac{1}{2} k^{\alpha\beta} Q_{.\alpha\beta j} \,, \\ & R_{1\lambda\mu 1,j} = \frac{1}{2} Q_{.\lambda\mu j} \,, \qquad C_{1\lambda\mu 1,j} = \frac{1}{2}\left(Q_{.\lambda\mu j} - \frac{1}{n-2} k_{\lambda\mu} (k^{\alpha\beta} Q_{.\alpha\beta j}) \right). \end{aligned} \tag{14}$$

Substituting (9) into (12), (13) and (14), we easily obtain

$$
\begin{aligned}
&S_{11} = (n-2)A, \quad R_{1\lambda\mu 1} = Ak_{\lambda\mu} + c_{\lambda\mu}, \quad C_{1\lambda\mu 1} = c_{\lambda\mu},\\
&S_{11,j} = (n-2)A_{.j}, \quad R_{1\lambda\mu 1,j} = A_{.j}k_{\lambda\mu}, \quad C_{1\lambda\mu 1,j} = 0,
\end{aligned} \tag{15}
$$

which, since $g^{11} = 0$, implies (i) and (ii).

Moreover, using (11), $R_{1\lambda\mu 1,j} = A'\delta^1_j k_{\lambda\mu}$ and $S_{11,j} = (n-2)A'\delta^1_j$, one can easily check that the only components of $\nabla^t R$ and $\nabla^t S$ not identically vanishing are

$$
\begin{aligned}
R_{1\lambda\mu 1,q_1\ldots q_t} &= A^{(t)}\delta^1_{q_1}\delta^1_{q_2}\ldots\delta^1_{q_t}k_{\lambda\mu}\,,\\
S_{11,q_1\ldots q_t} &= (n-2)A^{(t)}\delta^1_{q_1}\delta^1_{q_2}\ldots\delta^1_{q_t}\,,
\end{aligned} \tag{16}
$$

where the prime ((t) resp.) indicates the ordinary derivative (of order t resp.) with respect to x^1.

Assume now that (10) holds. Then, in view of (16), we get $\nabla^t R = 0$. Since, by (10) and (16), $\nabla^{t-1}R$ does not vanish, M is t-symmetric. Moreover, (16) yields $\nabla^t S = 0$, which, together with (10) and (16), shows that M is also Ricci t-symmetric.

This completes the proof of (iv).

Suppose that M is recurrent. Then, because of (15) and (2) (with R instead of T), we obtain $c_{\alpha\beta}k_{\lambda\mu} = c_{\lambda\mu}k_{\alpha\beta}$, which, since $k^{\alpha\beta}c_{\alpha\beta} = 0$ by assumption, implies $c_{\lambda\mu} = 0$, a contradiction. Thus, M cannot be recurrent. The second part of (iii) is an immediate consequence of $R_{1\lambda\mu 1,lm} = \frac{1}{A'}A''\delta^1_m R_{1\lambda\mu 1,l}$. This completes the proof.

Hence, we have

Corollary 1. *For each $n \geq 4$ and for each $t \geq 2$, there exist n-dimensional essentially conformally symmetric non-recurrent Ricci-recurrent metrics which are t-symmetric and Ricci t-symmetric.*

Remark 2. It is easy to prove that for the metric (8), we have

$$\text{index of } [g_{ij}] = \text{index of } [k_{\lambda\mu}] + 1\,,$$

the index of a symmetric matrix being understood as the number of negative entries in its diagonal form (for details see Remark 1 of [6]).

Remark 3. Obviously, if $Q = Ak_{\lambda\mu}x^\lambda x^\mu$ ($c_{\lambda\mu} = 0$) and $[k_{\lambda\mu}]$ has the properties stated in Example 1, then (15) yields

$$
\begin{gathered}
R_{1\lambda\mu 1} = Ak_{\lambda\mu}, \quad S_{11} = (n-2)A, \quad C_{1\lambda\mu 1} = 0,\\
S_{11,j} = (n-2)A'\delta^1_j, \quad R_{1\lambda\mu 1,l} = A'\delta^1_l k_{\lambda\mu}.
\end{gathered}
$$

Thus, in view of (10) and (16), we have

COROLLARY 2. *For each $n \geq 4$ and for each $t \geq 2$, there exist n-dimensional conformally flat recurrent metrics which are t-symmetric and Ricci t-symmetric.*

Since a parallel tensor vanishes if it vanishes at some point, Lemma 5 yields

COROLLARY 3. *A pseudo-Riemannian manifold of dimension $n \geq 3$ is projectively t-symmetric if and only if it is t-symmetric.*

Moreover, in view of Corollary 1, we get

COROLLARY 4. *For each $n \geq 4$ and for each $t \geq 2$, there exist n-dimensional essentially conformally symmetric Ricci-recurrent metrics which are projectively t-symmetric. Such metrics are necessarily t-symmetric.*

EXAMPLE 2. *Let $M = \{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^1 > 0 \text{ and } n \geq 4\}$ be endowed with the metric* (8), *where*

$$Q = (Ak_{\lambda\mu} + Bc_{\lambda\mu})x^\lambda x^\mu \,. \tag{17}$$

Assume moreover that $[k_{\lambda\mu}]$ and $[c_{\lambda\mu}]$ have the properties described in Example 1, *and A, B are smooth functions of x^1 only such that A does not identically vanish, $B \neq \text{const.}$, $B \neq 0$ everywhere and $A \neq cB$ $(c = \text{const.})$. Then*:

(i) *M is simple conformally recurrent.*

(ii) *M is Ricci-recurrent, non-recurrent and its scalar curvature vanishes.*

(iii) *If $B = a(x^1)^{t-1}$, where $t \geq 2$ and $a = \text{const.} \neq 0$, then $\nabla^t C = 0$ although $\nabla^{t-1} C \neq 0$ everywhere.*

(iv) *If B is as above and*

$$A = \frac{(t-1)(t+3)}{16(x^1)^2} \,,$$

then (M, g) admits a conformal change of metric $g \to \overline{g} = (\exp 2p)g$ such that $(M, \overline{g})$ is locally symmetric.

Proof. Substituting (17) into (12), (13), and (14) we easily obtain

$$\begin{aligned} &S_{11} = (n-2)A, \quad R_{1\lambda\mu 1} = Ak_{\lambda\mu} + Bc_{\lambda\mu}, \quad C_{1\lambda\mu 1} = Bc_{\lambda\mu}, \\ &S_{11,l} = (n-2)A_{.l}, \quad R_{1\lambda\mu 1,l} = A_{.l}k_{\lambda\mu} + B_{.l}c_{\lambda\mu}, \quad C_{1\lambda\mu 1,l} = B_{.l}c_{\lambda\mu}, \end{aligned} \tag{18}$$

which, because of $C_{1\lambda\mu 1,l} = (\log|B|)'\delta^1_l C_{1\lambda\mu 1} = a_l C_{1\lambda\mu 1}$, shows that M is conformally recurrent and its recurrence vector is given by $a_j = (\log|B|)'\delta^1_j$. Hence, in view of (18) and Lemma 1, M is simple conformally recurrent. Moreover, equations (18) and $g^{11} = 0$ show that M is Ricci-recurrent and that its scalar curvature vanishes everywhere.

Assume now that M is recurrent. Then, because of (2) and (18), we get $(BA'-AB')\delta_l^1 c_{\lambda\mu} = 0$. But this implies $A'-(B'/B)A = 0$ and, consequently, we must have $A = cB$ ($c =$ const.), a contradiction. Hence, M cannot be recurrent.

Using (11), (18) and $C_{1\lambda\mu 1,l} = B'\delta_l^1 c_{\lambda\mu}$ one can now easily check that the only components of $\nabla^t C$ not identically vanishing are

$$C_{1\lambda\mu 1,q_1\dots q_t} = B^{(t)}\delta_{q_1}^1\delta_{q_2}^1\dots\delta_{q_t}^1 c_{\lambda\mu}\,,$$

which, since $B = a(x^1)^{t-1}$ by assumption, completes the proof of (iii).

From (18) it follows that any smooth function of x^1 only (and in particular $p = \frac{1}{4}(t-1)\log x^1$) satisfies condition (e) of Lemma 2.

Thus, by Lemma 3, $(M,\overline{g})$ with $\overline{g} = (\exp 2p)g = (x^1)^{(t-1)/2}g$ is conformally recurrent.

On the other hand, the recurrence vector of (M,g) is given by $a_j = \frac{t-1}{x^1}\delta_j^1$, which, in view of Lemma 2, shows that $\overline{a}_j = 0$.

Hence, $(M,\overline{g})$ is conformally symmetric. It remains therefore to prove that the Ricci tensor of $(M,\overline{g})$ is parallel.

Since $p_i = \partial_i p = 0$ ($i = 2,\dots,n$), $g^{11} = 0$ and $S_{ij} = (n-2)A\delta_i^1\delta_j^1$, it follows that $p^r S_{ri}$ as well as $\Delta_1 p = p^r p_r$ and $\Delta_2 p = p^r{}_{,r}$ vanish everywhere. Thus, equations (5) and (6) imply

$$\begin{aligned}\overline{S}_{ij;k} = S_{ij,k} - 2p_k S_{ij} - p_i S_{jk} - p_j S_{ik} + (n-2)p_{i,jk}\\ + 4(n-2)p_i p_j p_k - 2(n-2)(p_i p_{j,k} + p_j p_{i,k} + p_k p_{i,j})\,,\end{aligned}$$

where the semicolon denotes covariant differentiation with respect to $\overline{g}$. Moreover, using (11) and $p_i = 0$ ($i = 2,\dots,n$) again, one can easily check that the only component of $\overline{\nabla}\,\overline{S}$ not identically vanishing is

$$\overline{S}_{11;1} = S_{11,1} - 4p_1 S_{11} + (n-2)p_{1,11} + 4(n-2)p_1^3 - 6(n-2)p_1 p_{1,1}\,.$$

But the last expression, in view of (11) and $S_{11} = (n-2)A$, takes the form

$$\overline{S}_{11;1} = (n-2)(p''' - 6p'p'' + 4(p')^3 - 4Ap' + A')\,. \tag{19}$$

Using now the definitions of p and A one can easily verify that $\overline{S}$ is parallel. This completes the proof.

Since in the above metric $R_{1\lambda\mu 1,q_1\dots q_t} = (A^{(t)}k_{\lambda\mu} + B^{(t)}c_{\lambda\mu})\delta_{q_1}^1\dots\delta_{q_t}^1$, Example 2 yields

COROLLARY 5. *For each $n \geq 4$ and for each $t \geq 2$, there exist n-dimensional non-recurrent simple conformally recurrent Ricci-recurrent non-t-symmetric metrics which are conformally t-symmetric and conformal to metrics with parallel curvature tensor.*

Remark 4. Assume that (M,g) has the properties described in Example 2. If $A = \frac{1}{4}(x^1)^{-2}$ and $B = (x^1)^{-2}$, then, as one can easily verify, (M,g)

is recurrent. Moreover, setting $p = -\frac{1}{2}\log x^1$, we get $\overline{a}_j = 0$ and $\overline{S}_{11;1} = 0$. Thus, we have

COROLLARY 6. *For each $n \geq 4$, there exist n-dimensional non-conformally flat recurrent metrics which are conformal to metrics with parallel curvature tensor* (cf. [10], Corollary 4.2).

Remark 5. Let $p = \frac{1}{4}(t-1)\log x^1$. Denote by g the metric described in (iv) of Example 2. Then $(M, \overline{g})$, where $\overline{g} = (\exp 2p)g$, is locally symmetric. Assume that q is a smooth function on M such that $(M, \overline{g}_1)$ with $\overline{g}_1 = (\exp 2q)g$ is conformally symmetric. Then, by Lemma 4, the condition $\overline{g} = (\exp 2(p-q))\overline{g}_1$ implies $q = p + c$, where $c = \text{const}$. Hence, by (19), $(M, \overline{g}_1)$ is locally symmetric too. This yields

COROLLARY 7. *For each $n \geq 4$ and for each $t \geq 2$, there exist n-dimensional non-recurrent simple conformally recurrent Ricci-recurrent non-t-symmetric metrics which are conformally t-symmetric and not conformal to any essentially conformally symmetric metric.*

Remark 6. Nickerson's result (Theorem C) is a consequence of Lemma 2. Indeed, the definition of a conformally recurrent manifold used in Nickerson's paper is given by (4) with $a_j \neq 0$ at some point. Since the considered manifold is not conformally flat by assumption, (4) yields $C \neq 0$ everywhere. Assume now that $(M, \overline{g})$ with $\overline{g} = (\exp 2p)g$ is conformally symmetric. Then, by Lemma 2, we must have $p^r p_r = 0$, which, since $\overline{a}_j = a_j - 4p_j$ and the metric is positive definite, leads immediately to the assertion.

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