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THE NUMBER OF COUNTABLE ISOMORPHISM TYPES OF COMPLETE EXTENSIONS OF THE THEORY OF BOOLEAN ALGEBRAS

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There is a conjecture of Vaught [17] which states: Without The Generalized Continuum Hypothesis one can prove the existence of a complete theory with exactly ω_1 nonisomorphic, denumerable models. In this paper we show that there is no such theory in the class of complete extensions of the theory of Boolean algebras. More precisely, any complete extension of the theory of Boolean algebras has either 1 or 2^ω nonisomorphic, countable models. Thus we answer this conjecture in the negative for any complete extension of the theory of Boolean algebras. In Rosenstein [15] there is a similar conjecture that any countable complete theory which has uncountably many denumerable models must have 2^ω nonisomorphic denumerable models, and this is true without using the Continuum Hypothesis.

This paper is an excerpt of the author's thesis, which was written under the guidance of Professor G. C. Nelson. A more detailed exposition of the material may be found there.

1. Preliminaries. Let L be a linear ordering. We define a derivative operation on L in two steps. First we identify any two elements which have a dense linear ordering between them, denoting the resulting ordering by L^a. Then in L^a we identify any two elements that have only a finite number of elements between them, denoting the result by L^1. This process can be iterated into the transfinite by taking an appropriate limit construction at limit ordinals. As an example, consider

$$\begin{aligned}
L &= (1+\eta+1)\cdot\omega+\omega^2\cdot(1+\eta+1)+1\,,\\
L^a &\cong \omega+\omega^2\cdot(1+\eta+1)+1\,,\\
L^1 &\cong 1+\omega\cdot(1+\eta+1)+1\cong\omega\cdot(1+\eta+1)+1\,,\\
(L^1)^a &\cong L^1\,,\quad L^2\cong 1+\eta+1\,,\\
(L^2)^a &\cong 1+1\,,\quad L^3\cong 1\,.
\end{aligned}$$

The language of Boolean algebras is the language with nonlogical symbols $\langle \cap, \cup, {}^{c}, 1, 0\rangle$. The theory T of Boolean algebras is the theory in this language with the usual axioms for Boolean algebras. The following classification of complete theories of Boolean algebras is due to Tarski [16] and Ershov [5] and can be found in Mead [9, Sections 3.2 and 3.3] as well. Let B be a Boolean algebra. The partial function $\Delta : B \to B$ is defined by

$$\Delta(x) = \sup\{y \in B : y \le x \text{ and } y \text{ is an atom of } B\} \quad \text{ for } x \in B\,.$$

By means of Δ we define an ideal of B,

$$I(B) = \{x \in B : \Delta(x) \text{ exists}\}\,.$$

We also define inductively a sequence of Boolean algebras B_k, $k \in \omega$:

$$B_0 = B, \qquad B_{k+1} = B_k / I(B_k)\,.$$

Definition 1. The *elementary characteristic* $\mathrm{EC}(B) = \langle p, q, r\rangle$ of a Boolean algebra B is defined as follows:

1. $\mathrm{EC}(B) = \langle 0, 0, 0\rangle$ if B is trivial, i.e. $0 = 1$.
2. $\mathrm{EC}(B) = \langle \omega, 0, 0\rangle$ if B_0 is not trivial for all $p \in \omega$.
3. If B_p is not trivial and B_{p+1} is trivial, then $\mathrm{EC}(B) = \langle p, q, r\rangle$, where:
   a) q is the number of atoms in B_p ($q = \omega$ if B_p has infinitely many atoms),
   b) $r = 0$ if B_p has no atomless elements, and $r = 1$ otherwise.

Theorem 2. *A theory T' is a complete extension of the theory T of Boolean algebras iff there is an ordered triple $\langle p, q, r\rangle$ with $p, q \in \omega \cup \{\omega\}$ such that every model of T' has elementary characteristic $\langle p, q, r\rangle$.*

From Theorem 2 it follows that two Boolean algebras are elementarily equivalent iff they have the same elementary characteristic. Thus the notation $B \equiv \langle p, q, r\rangle$ for $\mathrm{EC}(B) = \langle p, q, r\rangle$ is well defined, and will be frequently used.

As examples of Boolean algebras we will use interval algebras. In the rest of this paper it will be assumed that L is a linear ordering with a first element f and a last element t. We define the *interval algebra* of L, denoted by $D(L)$, to be the subalgebra of the power set of $[f, t)$ generated by $\{[x, y) : x, y \in L \text{ and } x \le y\}$ with the usual operations on sets of union, intersection, and relative complement in $[f, t)$. It is easily shown that $D(L)$ is a Boolean algebra with ordered basis L.

Now we define a derivative operation for Boolean algebras, which, in the case of interval algebras, will be the same as taking derivatives on the linearly ordered basis as above. This operation on Boolean algebras was introduced and studied by Nelson in [14, Chapter 2]. Its connections with the derivatives of L^a and interval algebras were first stated there. The idea of making such a connection appears in Feiner [6], but for a different notion of derivative.

Let B be a Boolean algebra, and let

$$\begin{aligned} A_0(B) &= (0) \quad \text{(the zero ideal)}\,, \\ At_0(B) &= \{a \in B : a \text{ is an atom of } B\}\,, \\ Al_0(B) &= \{a \in B : a \text{ is an atomless element of } B\}\,, \\ A_1(B) &= (At_0(B) \cup Al_0(B)) \\ &\qquad\qquad\qquad \text{(the ideal generated by } At_0(B) \cup Al_0(B)), \\ B_1 &= B/A_1(B)\,. \end{aligned}$$

Suppose that $A_\beta(B)$ and $B^\beta = B/A_\beta(B)$ have been defined for $\beta < \alpha$. Then there are natural homomorphisms $h_\beta : B \to B/A_\beta(B)$. For $\beta < \alpha$ we let

$$At_\beta(B) = \{b \in B : h_\beta(b) \text{ is an atom of } B^\beta\}\,,$$

$$Al_\beta(B) = \{b \in B : h_\beta(b) \text{ is an atomless element of } B^\beta\}\,.$$

Then we define

$$A_\alpha(B) = \Big(\bigcup_{\beta<\alpha} At_\beta(B) \cup \bigcup_{\beta<\alpha} Al_\beta(B)\Big)\,, \qquad B^\alpha = B/A_\alpha(B)\,,$$

$$h_\alpha : B \to B/A_\alpha(B) \text{ is the natural homomorphism}\,.$$

We begin by applying this approach to the problem of computing the elementary characteristics of a Boolean algebra.

THEOREM 3. *Let L be a linear ordering with the property that every infinite interval of the form $[a, b)$ contains an infinite dense subinterval. Then*

$$D(L)_1 = D(L)/I(D(L)) = (D(L))^1 \cong D(L^1)\,.$$

2. The number of countable isomorphism types. Now we state the main result of this paper.

THEOREM 4. *For Γ a complete extension of the theory T of Boolean algebras, Γ has either exactly one countable model* (*up to isomorphism*), *or 2^ω nonisomorphic*, *countable models.*

We will give the proof in cases depending on the elementary characteristic of Γ. In each case we will either demonstrate that the theory is ω-categorical, or give examples of 2^ω nonisomorphic, countable models of Γ.

The first case is for the elementary characteristics $\langle 0, \omega, 0\rangle$, and $\langle 0, \omega, 1\rangle$. In [14, Theorem 2.2.8] Nelson has proved the result for this case and has given the explicit examples we use for the characteristics $\langle 0, \omega, 0\rangle$, and $\langle 0, \omega, 1\rangle$. It is known that there are 2^ω nonisomorphic countable Boolean algebras, and the usual examples given are similar to those in [14] and have characteristics $\langle 0,\omega,0\rangle$, or $\langle 0,\omega,1\rangle$. Here the other complete extensions of the theory of Boolean algebras are considered. Apparently the result was not known up to now for these theories.

For each $f \in 2^{\omega-\{0\}}$ for which f is not identically zero let $L(f)$ be the linear ordering given by

$$L(f) = \Big(\sum_{i\in\omega-\{0\}} f(i)\cdot(\omega^i\cdot\eta+1)\Big),$$

where $0\cdot(\omega^i\cdot\eta+1)=0$. We consider the interval algebras $D(1+L(f)+1)$. For each $f \in 2^{\omega-\{0\}}$, $D(1+L(f)+1)$ is infinite and atomic and so $D(1+L(f)+1) \equiv \langle 0,\omega,0\rangle$. Furthermore, suppose f is different from g and let both be elements of $2^{\omega-\{0\}}$. Then there is an $i \in \omega-\{0\}$ such that $f(i)\neq g(i)$. Suppose $f(i)=0$ and $g(i)=1$. Then $D((1+L(f)+1)^i)$ is infinite atomic while $D((1+L(g)+1)^i)$ contains an atomless element. Thus

$$\begin{aligned}(D(1+L(f)+1))^i &\cong D((1+L(f)+1)^i)\\ &\ncong D((1+L(g)+1)^i) \cong (D(1+L(g)+1))^i\,.\end{aligned}$$

So there is a distinct model for each $f\in 2^{\omega-\{0\}}$ and therefore, there must be 2^ω nonisomorphic, countable models of T with elementary characteristic $\langle 0,\omega,0\rangle$.

To give examples of countable models with other elementary characteristics, we let L_0 be the linear ordering defined by

$$L_0=(1+\eta+1)\cdot\zeta\,.$$

We state a useful preliminary lemma, whose proof can be found in Iverson [7].

LEMMA 5. $D(1+L_0\cdot L+1)_1 \cong D(1+L_0\cdot L+1)^1 \cong D(1+L+1)$. *Furthermore, if* $D(1+L+1)\equiv\langle n,p,q\rangle$, *then* $D(1+L_0\cdot L+1)\equiv\langle n+1,p,q\rangle$.

Inductively, suppose there are 2^ω nonisomorphic, countable models of T with elementary characteristic $\langle n,\omega,0\rangle$ for some $n\in\omega$ and suppose that the models have linearly ordered bases $1+A_n(f)+1$ for $f\in 2^{\omega-\{0\}}$. Then consider the linear ordering $1+L_0\cdot A_n(f)+1$ for each $f\in 2^{\omega-\{0\}}$. We have

$$D(1+L_0\cdot A_n(f)+1)\equiv\langle n+1,\omega,0\rangle$$

by Lemma 5. Furthermore, for $f\neq g$ the inductive hypothesis yields that

$$D(1+A_n(f)+1)\ncong D(1+A_n(g)+1)\,.$$

So, since

$$(D(1+L_0\cdot A_n(f)+1))^1 \cong D((1+L_0\cdot A_n(f)+1)^1) \cong D(1+A_n(f)+1)$$

and the same holds for $1 + A_n(g) + 1$, it must be true that

$$D(1 + L_0 \cdot A_n(f) + 1) \ncong D(1 + L_0 \cdot A_n(g) + 1)\,.$$

Finally, it is clear that all of these Boolean algebras are countable, since all of their linearly ordered bases are countable. So there are 2^ω nonisomorphic, countable models of T with elementary characteristic $\langle n+1, \omega, 0\rangle$ and linearly ordered bases

$$1 + A_{n+1}(f) + 1 = 1 + L_0 \cdot A_n(f) + 1 \qquad \text{for } f \in 2^{\omega - \{0\}}\,.$$

By induction it follows that for any $n \in \omega$ there are 2^ω nonisomorphic, countable models of T with elementary characteristic $\langle n, \omega, 0\rangle$.

Now we consider the elementary characteristics of the form $\langle n, \omega, 1\rangle$. For $n = 0$ we use the interval algebras of the form $D(1 + L(f) + 1 + \eta + 1)$ for $f \in 2^{\omega-\{0\}}$. As before, if $f \neq g$, then $f(i) \neq g(i)$ for some $i \geq 1$. Then one of $D((1+L(f)+1+\eta+1)^i)$ and $D((1+L(g)+1+\eta+1)^i)$ is infinite atomic and the other contains an atomless element, so

$$D(1 + L(f) + 1 + \eta + 1) \ncong D(1 + L(g) + 1 + \eta + 1)\,.$$

Furthermore, every element of $D(1+L(f)+1+\eta+1)$ is a finite sum of atoms and atomless elements so $\mathrm{EC}(D(1+L(f)+1+\eta+1)) = \langle 0, \omega, 1\rangle$. Therefore there are 2^ω nonisomorphic, countable Boolean algebras with elementary characteristic $\langle 0, \omega, 1\rangle$. Now we proceed inductively as before, using L_0 to antidifferentiate. The details are completely analogous. It follows that for any $n \in \omega$ there are 2^ω nonisomorphic, countable Boolean algebras with elementary characteristic $\langle n, \omega, 1\rangle$.

Next we consider the characteristic $\langle n, m, 0\rangle$ for $n, m \in \omega$. For any $m \in \omega$ the interval algebras of the form $D(1 + \eta + L(f) + 1)$ give 2^ω nonisomorphic, countable models of T, which can be shown to have elementary characteristic $\langle 1, m, 0\rangle$ (see Iverson [7] for details). Then, inductively, using L_0 to antidifferentiate and applying Lemma 5, for any $n, m \in \omega,\ n \neq 0$, we get 2^ω nonisomorphic, countable models of T with elementary characteristic $\langle n, m, 0\rangle$.

To show that for any $n, m \in \omega$ there are 2^ω nonisomorphic, countable models of T with elementary characteristic $\langle n, m, 1\rangle$, we use the interval algebras

$$D(1 + (\eta + L(f) + 1 + L_0 \cdot \eta) \cdot m + 1)$$

for the ground step of the induction. These have characteristic $\langle 1, m, 1\rangle$. The arguments are exactly analogous to those above.

For elementary characteristic $\langle \omega, 0, 0\rangle$ we consider the linear ordering

$$1 + (L_0 + L_0 \cdot L_0 + L_0 \cdot L_0 \cdot L_0 + \ldots) \cdot L(f) + 1$$

for $f \in 2^{\omega - \{0\}}$, which we call $A(f)$. We have $D(A(f)) \equiv \langle \omega, 0, 0 \rangle$ by Theorem 3, since

$$D(A(f))_n \cong D(A(f))^n \cong D(A(f)^n) \cong D(A(f))$$

for any $n \in \omega$. Applying Theorem 3 gives 2^ω nonisomorphic, countable models of T with elementary characteristic $\langle \omega, 0, 0 \rangle$.

Finally, we consider the cases where the models have only finitely many atoms. It is easily seen that for any $m \in \omega$ there is only one countable isomorphism type of T with elementary characteristic $\langle 0, m, 0 \rangle$, as well as there is only one countable isomorphism type of T with elementary characteristic $\langle 0, 0, 1 \rangle$, and it follows that for any $m \in \omega$ there is only one countable isomorphism type of T with elementary characteristic $\langle 0, m, 1 \rangle$.

If a countable model of T has only finitely many atoms, then any other countable model of T with the same complete theory is isomorphic to it. Otherwise there are 2^ω nonisomorphic countable models with the same theory. So the theorem is proved and Vaught's conjecture is answered in the negative for any complete extension of the theory of Boolean algebras, without the use of the Continuum Hypothesis. On the other hand, we have given more evidence to support the conjecture that a complete countable theory with uncountably many nonisomorphic denumerable models has 2^ω nonisomorphic denumerable models (without using the Continuum Hypothesis).

Finally, by applying results of Burris and Nelson, it readily follows that this result can be extended to all countable primal algebras using the notion of bounded Boolean powers (see Iverson [7] for details).

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