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INTEGRAL EQUATIONS OF CONVOLUTION TYPE WITH POWER NONLINEARITY *

BY

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In the cone Γ of nonnegative continuous functions on $\mathbb{R}_+ = [0,\infty)$ the solvability of the nonlinear equation $u^\alpha = k * u + f$, $\alpha > 1$, with the kernel $k \in \Gamma$, is considered. That equation appears in applications of filtration theory. The present paper is connected with getting rid of the assumption $k(0+) > 0$, which is essentially utilized when considering the solvability of the above equation in W. Okrasiński's papers. According to the singularity of the kernel as $x \to 0$, two cases are considered:

(a) $k'(x)$ nondecreasing, $k(0) = 0$, $k'(0) > 0$;
(b) $k(x) = px^\nu + o(x^\nu)$, $x \to 0$, $p > 0$, $\nu > -1$.

The existence of a solution (nontrivial for $f(x) \equiv 0$) and its uniqueness are proved. In case (a) the kernel is smoother than in (b) and the results are more complete. Moreover, under assumptions (a), (b), the stability of solutions with respect to the perturbations of k, α and f is studied. Finally, the case $0 < \alpha < 1$ is considered. It is shown that in this case the equation may have at most one solution in Γ if $k, f \in \Gamma$. It is also shown, via the method of monotone operators, that for particular values of $\alpha = p - 1$, $p = 2n/(2n-1)$, $n = 1, 2, 3, \ldots$, the equation has a unique solution in the (real) space $L_p(\mathbb{R}_+)$ for $f \in L_{2n}(\mathbb{R}_+)$ and for the kernel $k \in L_1(\mathbb{R}_+) \cap L_n(\mathbb{R}_+)$ satisfying the condition $\mathrm{Re}\,\widehat{k}(x) \le 0$, $0 \le x < \infty$, where $\widehat{k}$ denotes the Fourier transform of k.

0. Introduction. It was shown in [5] and [9]–[11] that the nonlinear differential Boussinesq equation (cf. [3], [12])

$$(hh_r)_r + r^{-1}hh_r = h_t, \quad h = h(r,t)\,,$$

which describes the process of infiltration of a fluid from a cylindrical reservoir into an isotropic, homogeneous, porous medium can be reduced to a

* This survey was written mainly on the basis of papers of W. Okrasiński, S. N. Askhabov, N. K. Karapetyants and other authors.

nonlinear integral equation of convolution type of the form

$$u^{\alpha}(x) = \int_0^x k(x-t)u(t)\,dt, \qquad x > 0\,, \tag{0.1}$$

where the kernel $k(x)$ is, by physical reasons, a nonnegative and nondecreasing function. One looks for solutions of (0.1) in the cone Γ of nonnegative, continuous functions defined on the nonnegative half-line $x \geq 0$. W. Okrasiński showed that under the assumption $k(0+) = q > 0$ equation (0.1) has for $\alpha > 1$ a unique solution in the subclass $\Gamma_0 \subset \Gamma$ of functions which are positive for $x > 0$. The solution belongs to the cone interval

$$\left(\frac{\alpha-1}{\alpha}qx\right)^{1/(\alpha-1)} \leq u(x) \leq \left(\frac{\alpha-1}{\alpha}\int_o^x k(t)\,dt\right)^{1/(\alpha-1)}$$

and may be found by the method of successive approximations in a suitably chosen metric. The results were also generalized to the case of a nonhomogeneous linear part:

$$u^{\alpha}(x) = \int_0^x k(x-t)u(t)\,dt + f(x)\,. \tag{0.2}$$

One of the main assumptions in [9]–[10] is $k(0+) > 0$, since the convergence of successive approximations depends there on $1/k(0+)$.

In the present paper we discuss the problem of existence of nonnegative solutions of (0.1) and (0.2) without assuming $k(0+) > 0$.

The paper consists of five sections. In Section 1 we assume $k(0) = 0$, $k'(0) = p > 0$, which allows us to get an exact lower bound for solutions in the class Γ_0. The importance of this estimate follows from a special role it plays in estimating the rate of convergence. In particular, a new metric (cf. [9], [10]) in the class of solutions is introduced. It is shown that equation (0.1) has a unique solution in the cone Γ_0. A scheme is also proposed for the construction of the solution.

In Section 2 a generalization of results of [10] is given: one assumes that the kernel k has the form $k(x) = px^{\nu} + l(x)$, $p > 0$, $l(x) \geq 0$, $\nu \geq 0$, $x^{-\nu}l(x) \to 0$ as $x \to 0$. Moreover, the assumptions concerning the smoothness of k are less restrictive, which in contrast to Section 1 gives a possibility to study not necessarily monotonic solutions.

In Section 3 we consider the dependence of solutions of equation (0.1) on the kernel k and on the exponent α.

In Section 4 equation (0.2) is considered. Here for $\alpha > 1$ the existence of nonnegative solutions under assumptions made in Sections 1 and 2 is studied together with the question of the dependence of solutions, this time, on perturbations of f. The arguments used permit the assumptions concerning f to be essentially weakened in comparison with [10].

Finally, in Section 5 the case $0 < \alpha < 1$ is considered. Here the picture of solvability of (0.1) changes in comparison with the case $\alpha > 1$. For example, it turns out that equation (0.1) has only a trivial solution in the cone Γ whereas equation (0.2) may have only one solution. Moreover (see part B of Section 5), the questions of existence and uniqueness of solutions to equations (0.1)–(0.2) in real $L_p(\mathbb{R}_+)$ spaces are considered for some particular values of α.

The results of this paper were partially announced in [1]–[2] and [6].

1. Case $k(0) = 0$, $k'(0) > 0$. We assume throughout this section that

$$k'(x) \text{ is nondecreasing, } k(0) = 0 \text{ and } k'(0) = p > 0\,. \tag{1.1}$$

Denote by Γ the cone (see [7]) of nonnegative continuous functions defined for $x \geq 0$. It is easy to see that if $u \in \Gamma$ is a solution of equation (0.1) then so are the translations

$$u_\delta(x) = \begin{cases} u(x-\delta) & \text{for } x > \delta, \\ 0 & \text{for } 0 < x \leq \delta, \end{cases}$$

for any $\delta > 0$. To exclude the resulting nonuniqueness we introduce the following class of functions:

$$\Gamma_0 = \{u : u \in \Gamma,\ u(0) = 0,\ u(x) > 0 \text{ for } x > 0\}\,.$$

LEMMA 1.1. *If $u \in \Gamma_0$ is a solution of* (0.1) *then it is nondecreasing and*

$$C(\alpha)x^{2/(\alpha-1)} \leq u(x) \leq \left(\frac{\alpha-1}{\alpha}\int_0^x k(t)\,dt\right)^{1/(\alpha-1)}, \tag{1.2}$$

where

$$C(\alpha) = \left(\frac{k'(0)(\alpha-1)^2}{2\alpha(\alpha+1)}\right)^{1/(\alpha-1)}.$$

Proof. Since, by assumption, k is nondecreasing, so is u (see [10]) and

$$u'(x) = \frac{1}{\alpha}u^{1-\alpha}(x)(k' * u)(x)\,. \tag{1.3}$$

The last formula implies that $u'(x)$ is continuous for $x > 0$ and $u''(x)$ exists. We shall find the lower bound in (1.2). We get from (1.3)

$$(u^\alpha)'' = pu + k'' * u \geq pu\,.$$

Putting $u^\alpha(x) = v(x)$ and $z = v'$ we get $zz' \geq pv^{1/\alpha}$, which implies $z \geq \sqrt{2p\alpha/(\alpha+1)}\,v^{(\alpha+1)/2\alpha}$. Integrating the last inequality we get the lower bound in (1.2).

To find the upper bound we make use of (1.3):

$$(u^\alpha(x))' = \alpha u^{\alpha-1}(x)u'(x) = \int_0^x k(x-t)u'(t)\,dt \le k(x)u(x)\,.$$

Since $u \in \Gamma_0$, we hence get

$$\frac{\alpha}{\alpha-1}(u^{\alpha-1}(x))' \le k(x)\,.$$

The desired upper bound follows by integrating the last inequality.

Define

$$\Omega_b = \{u : u \in C[0,b] \text{ and satisfies (1.2)}\}\,.$$

Put $\Omega_\infty = \bigcup_{b>0} \Omega_b$ and

$$(Tu)(x) = \left(\int_0^x k(x-t)u(t)\,dt\right)^{1/\alpha}.$$

LEMMA 1.2. *The operator T maps the set Ω_b into itself.*

Proof. Let $u \in \Omega_b$. It is easy to see that $(Tu)(x)$ satisfies the upper estimate in (1.2). To check the lower estimate it suffices to integrate by parts and note that $k'(x) \ge p$, which gives

$$\begin{aligned}[(Tu)(x)]^\alpha &\ge \int_0^x k(x-t)C(\alpha)t^{2/(\alpha-1)}\,dt \\ &= \frac{\alpha-1}{\alpha+1}C(\alpha)\int_0^x t^{(\alpha+1)/(\alpha-1)}k'(x-t)\,dt \ge (C(\alpha)x^{2/(\alpha-1)})^\alpha\,.\end{aligned}$$

Lemma 1.2 is proved.

We introduce in Ω_b a metric by the formula (cf. [9, 10])

$$\varrho_b(u_1,u_2) = \sup_{0<x\le b}\frac{|u_1(x)-u_2(x)|}{e^{\beta x}x^{2/(\alpha-1)}}, \quad \beta = \frac{1}{p}\sup_{a\le x\le b}\frac{k'(x)-p}{x}\,, \tag{1.4}$$

where $b < \infty$ and $a > 0$ is chosen so that $k'(a) < \alpha k'(0)$.

Define

$$g(x) = e^{\beta x}x^{2/(\alpha-1)}, \quad r(x) = x^{2/(\alpha-1)}\,.$$

We check, as in [9], that Ω_b with the metric ϱ_b is a complete metric space. We shall show that the operator T is a contraction. Applying the Lagrange theorem as in [10] we have

$$\varrho_b(Tu_1,Tu_2) \le \frac{1}{\alpha}\sup_{0<x\le b}\frac{|[k*(u_2-u_1)](x)|}{\{\min([Tu_2(x)]^\alpha,[Tu_1(x)]^\alpha)\}^{(\alpha-1)/\alpha}g(x)}\,,$$

from which, by using the lower estimate for $(Tu)(x)$ of Lemma 1.2, we get

$$\varrho_b(Tu_1,Tu_2) \le \frac{1}{\alpha}[C(\alpha)]^{1-\alpha} \sup_{0<x\le b} \frac{|[k*(u_2-u_1)](x)|}{x^2e^{\beta x}r(x)} \tag{1.5}$$

and since $|[k*(u_2-u_1)](x)| \le (k*[e^{\beta t}r(t)])(x)\varrho_b(u_1,u_2)$, we have

$$\varrho_b(Tu_1,Tu_2) \le \frac{1}{\alpha}[C(\alpha)]^{1-\alpha} \sup_{0<x\le b} \frac{(k*[e^{\beta t}r(t)])(x)}{x^2e^{\beta x}r(x)}\varrho_b(u_1,u_2)\,. \tag{1.5$'$}$$

In what follows the following lemma will be needed.

LEMMA 1.3. *For any $a>0$ and $x\in[0,b]$*

$$k(x)e^{-\beta x} \le xk'(x)e^{-\beta x} \le xk'(a)\,. \tag{1.6}$$

Proof. Applying (1.1) we get $k(x)\le xk'(x)$, which gives the left-hand inequality in (1.6). To prove the other inequality it suffices to apply Lemma 7 of [9] to the function $k'(x)$.

Let us return to the estimate (1.5$'$). Lemma 1.3 implies

$$e^{-\beta x}(k*[e^{\beta x}r(x)]) \le k'(a)\int_0^x (x-t)r(t)\,dt\,.$$

Applying this inequality to (1.5$'$) we get

$$\varrho_b(Tu_1,Tu_2) \le \frac{k'(a)}{\alpha[C(\alpha)]^{\alpha-1}}\varrho_b(u_1,u_2)\sup_{0<x\le b}\frac{\int_0^x (x-t)r(t)\,dt}{x^2r(x)}\,.$$

The supremum is equal to $(\alpha-1)^2/2\alpha(\alpha+1)$, hence

$$\varrho_b(Tu_1,Tu_2) \le \frac{1}{\alpha}\frac{k'(a)}{k'(0)}\varrho_b(u_1,u_2)\,. \tag{1.7}$$

Since for any $b>0$ there exists $a>0$ such that $k'(a)<\alpha k'(0)$, the operator T is a contraction and the following is true.

THEOREM 1.1. *The nonlinear equation* (0.1) *of convolution type has a unique solution in Γ_0 as well as in any Ω_b, $b\le\infty$. It may be found by the method of successive approximations.*

Proof. A solution $u(x)$ sought in the class Γ_0 satisfies automatically the inequalities (1.2), which implies $u\in\Omega_\infty$.

On the other hand, the coefficient $k'(a)(\alpha k'(0))^{-1}$ in (1.7) does not depend on b, which means that the unique solution obtained by the method of successive approximations belongs to Ω_b for any $b>0$, and this implies $u\in\Omega_\infty$. Therefore it is unique in Γ_0 too.

Remark 1.1. The lower estimate in (1.2) cannot be improved: this follows from the fact that the function $u(x) = C(\alpha)x^{2/(\alpha-1)}$ is a solution of (0.1) with the kernel $k(x) = px$ satisfying all assumptions of Section 1.

The contraction principle guarantees that the solution u can be obtained as $\lim_{n\to\infty} T^n v$, where v is any element of Ω_b. Let us choose v to be equal to $F(x) = C(\alpha)r(x)$ appearing in (1.2) as well as in the definition of Ω_b. It is convenient since $F \in \Omega_b$ and in consequence $T^n F \to u$ on $[0, b]$ for any b. The rate of convergence is given by

LEMMA 1.4. *Let* $u(x) = \lim_{n\to\infty}(T^n F)(x)$. *Then*

$$\varrho_b(T^n F, u) \le \frac{q^n}{1-q}\frac{[C(\alpha)]^{2-\alpha}}{\alpha}\sup_{0<x\le b}\frac{\int_0^x [k(t)-pt]\,dt}{x^2 e^{\beta x}}$$

where $q = k'(a)(\alpha k'(0))^{-1} < 1$.

The proof follows by immediate calculations.

EXAMPLE 1.1. Assume that $k(x)$ is of the form $k(x) = px + \gamma x^\mu$, $\mu > 1$, $\gamma > 0$. In this case

$$\int_0^x (k(t) - pt)\,dt = \frac{\gamma}{\gamma+1}x^{\mu+1}.$$

If b is so large that $(\mu-1)/\beta \in [0, b]$, then the maximum of the function $z(x) = \gamma(\mu+1)^{-1}x^{\mu-1}e^{-\beta x}$, $x \in [0, b]$, is equal to

$$z_{\max} = z\left(\frac{\mu-1}{\beta}\right) = \frac{\gamma}{\mu+1}\left(\frac{\mu-1}{\beta}\right)^{\mu-1} e^{1-\mu}.$$

Hence

$$\varrho_b(T^n F, u) \le \frac{q^n}{1-q}\frac{1}{\alpha}[C(\alpha)]^{2-\alpha} z_{\max}.$$

In particular, for $\alpha = 2$, $\gamma = 1$ and $\mu = 2$ we have

$$\varrho_b(T^n F, u) \le \frac{k'(0)}{3\beta[2k'(0) - k'(a)]}\left[\frac{k'(a)}{2k'(0)}\right]^n.$$

2. Case $k(x) = px^\nu + o(x^\nu)$. Assume that the kernel has the form

$$k(x) = px^\nu + l(x), \quad p > 0,\ \nu \ge 0, \tag{2.1}$$

where

$$l \in \Gamma \quad \text{and} \quad x^{-\nu} l(x) \to 0 \ \text{ as } \ x \to 0. \tag{2.2}$$

Note that for $\nu = 0$ the results of this section are similar to the results of [9, 10] and for $\nu = 1$ to those of Section 1.

LEMMA 2.1. *If* $k(x) = px^\nu$, $\nu > -1$, *then the function*

$$F(x) = \gamma x^{(\nu+1)/(\alpha-1)}, \quad \gamma = \left[pB\left(\nu+1, \frac{\nu+\alpha}{\alpha-1}\right)\right]^{1/(\alpha-1)}, \tag{2.3}$$

where B *is Euler's Beta function*, *is a solution of* (0.1).

The proof is by immediate calculations.

It is easy to check that

$$F(x) \le (TF)(x) \text{ and } u(x) \le \Big(\int_0^x [pt^\nu + l(t)]\,dt\Big)^{1/(\alpha-1)} \equiv G(x)\,, \tag{2.4}$$

if $u \in \Gamma_0$ is a solution of (0.1).

Define (cf. [2])

$$\Omega_b' = \{u : u \in C[0,b],\ F(x) \le u(x) \le G(x)\}\,.$$

Similarly to Section 1, we introduce in Ω_b' $(b < \infty)$ a metric by the formula

$$\varrho_b(u_1,u_2) = \sup_{0<x\le b} \frac{|u_1(x) - u_2(x)|}{e^{\beta x} r(x)}\,,$$

where $r(x) = x^{(\nu+1)/(\alpha-1)}$ and $\beta > 0$ is given by (2.6) below. It is easy to check that Ω_b' with the metric ϱ_b is a complete metric space.

LEMMA 2.2. *For any* $x \in [0,b]$ *the inequality*

$$k(x)e^{-\beta x} \le (p+\varepsilon)x^\nu \tag{2.5}$$

is true for any $\varepsilon > 0$ *and* $\beta = \beta(\varepsilon, b)$ *given by* (2.6) *below.*

P r o o f. We have to show the inequality

$$[px^\nu + l(x)]e^{-\beta x} \le px^\nu + \varepsilon x^\nu\,,$$

which holds if $l(x)e^{-\beta x} \le \varepsilon x^\nu$ for $x \in [0,b]$. For a given ε a number $a \in (0,b]$ may be found such that $x^{-\nu}l(x) < \varepsilon$ for all $x \in (0,a)$. It now suffices to put

$$\beta = \frac{1}{a} \ln \frac{\max_{a\le x\le b} l(x)}{\varepsilon a^\nu}\,. \tag{2.6}$$

LEMMA 2.3. *The operator* $T : \Omega_b' \to \Omega_b'$ *defined as before is a contraction* (*for sufficiently small* ε) *and for every* $\varepsilon > 0$

$$\varrho_b(Tu_1, Tu_2) \le \frac{p+\varepsilon}{\alpha p}\varrho_b(u_1,u_2)\,.$$

The proof is similar to that of (1.7), this time making use of Lemmas 2.1 and 2.2.

We have $(p+\varepsilon)/\alpha p < 1$ for sufficiently small ε; this yields

THEOREM 2.1. *Equation* (0.1) *with kernel of the form* (2.1) *has a unique continuous solution defined on* $[0,\infty)$ *satisfying the inequality*

$$u(x) \ge \gamma x^{(\nu+1)/(\alpha-1)}$$

with γ *given by* (2.3)

Remark 2.1. The results of this section remain true for $\nu > -1$ and for $l(x) \ge 0$ bounded on $[0,\infty)$ such that $x^{-\nu} l(x) \to 0$ as $x \to 0$.

3. Dependence of solutions on k and α. In this section we consider the problem of dependence of solutions to (0.1) on perturbations of k and of α. We begin with the former.

THEOREM 3.1. *Let* $u \in \Gamma_0$ *be a solution of the equation* $u^\alpha = k_1 * u$ *where* k_1 *is a kernel satisfying* (1.1). *If* $v \in \Gamma_0$ *is a solution of* $u^\alpha = k_2 * u$ *with kernel* k_2 *satisfying* (1.1) *then for* $b > 0$

$$\sup_{0\le x\le b} |u(x)-v(x)| \le C \sup_{0<x\le b} \frac{\int_0^x |k_1(t)-k_2(t)|\,dt}{x^{2\alpha/(\alpha-1)} e^{\beta_1 x}}, \tag{3.1}$$

where

$$C = C(\alpha,k_1,k_2,b) = \frac{e^{\beta_1 b} b^{2/(\alpha-1)} 2\alpha(\alpha+1)}{[\alpha k_1'(0) - k_1'(a)](\alpha-1)^2} \left[\frac{\alpha-1}{\alpha} \int_0^b k_2(t)\,dt\right]^{1/(\alpha-1)},$$

$$\beta_1 = \tfrac{1}{p} \sup_{a\le x\le b} \frac{k_1'(x) - p}{x},$$

and $a > 0$ *is chosen so that* $k_1'(a) < \alpha k_1'(0)$.

Proof. Since $\varrho_b(u,v) = \varrho_b(T_1 u, T_2 v) \le \varrho_b(T_1 u, T_1 v) + \varrho_b(T_1 v, T_2 v)$, where $(T_i u)(x) = [(k_i * u)(x)]^{1/\alpha}$, by (1.7) we obtain

$$\varrho_b(T_1 u, T_1 v) \le \frac{1}{\alpha} \frac{k_1'(a)}{k_1'(0)} \varrho_b(u,v). \tag{3.2}$$

We also have a relation similar to (1.5):

$$\varrho_b(T_1 v, T_2 v) \le \frac{1}{\alpha} [C(\alpha)]^{1-\alpha} \sup_{0<x\le b} \frac{|([k_1-k_2]*v)(x)|}{x^2 e^{\beta_1 x} r(x)}$$

$$\le \left[\frac{\alpha-1}{\alpha} \int_0^b k_2(t)\,dt\right]^{1/(\alpha-1)} \frac{1}{\alpha} [C(\alpha)]^{1-\alpha} \sup_{0<x\le b} \frac{\int_0^x |k_1(t)-k_2(t)|\,dt}{x^2 e^{\beta_1 x} r(x)}.$$

Combining these two estimates we obtain (3.1), and the theorem is proved.

Assume now that the kernel k satisfying condition (1.1) is fixed and u, v are solutions of the equations $u^{\alpha_i} = k * u$, $i = 1, 2$, respectively. Let $\varrho_{b,i}$, T_i be the corresponding metric and operator respectively and let, to be definite, $1 < \alpha_1 < \alpha_2$. Then $\varrho_{b,2}(u, v) = \varrho_{b,2}(T_1 u, T_2 v) \le \varrho_{b,2}(T_1 u, T_2 u) + \varrho_{b,2}(T_2 u, T_2 v)$ and by applying (3.2) we get

$$\left(1 - \frac{1}{\alpha_2}\frac{k'(a)}{k'(0)}\right)\varrho_{b,2}(u, v) \le \varrho_{b,2}(T_1 u, T_2 u)\,.$$

In order to estimate the right-hand side we use the inequality $1 - r^x \le x \ln(1/r)$, valid for $r > 0$, $x > 0$. We have

$$\begin{aligned} \text{(3.3)} \qquad & \varrho_{b,2}(T_1 u, T_2 u) \\ & \le \sup_{0<x\le b} \frac{[(k*u)(x)]^{1/\alpha_2}\left(\frac{1}{\alpha_1} - \frac{1}{\alpha_2}\right)\left|\ln\frac{1}{(k*u)(x)}\right|}{x^{2/(\alpha_2-1)}e^{\beta x}} \\ & = \left(1 - \frac{\alpha_1}{\alpha_2}\right)\sup_{0<x\le b}\frac{[u(x)]^{\alpha_1/\alpha_2}|\ln u(x)|}{x^{2/(\alpha_2-1)}e^{\beta x}}\,. \end{aligned}$$

Applying the inequality (1.2) from Lemma 1.1 we get

$$\frac{[u(x)]^{\alpha_1/\alpha_2}}{x^{2/(\alpha_2-1)}} \le \left[\frac{\alpha_1 - 1}{\alpha_1}\frac{k(x)}{x}\right]^{\alpha_1/(\alpha_1-1)\alpha_2} x^{2(\alpha_2-\alpha_1)/(\alpha_1-1)\alpha_2(\alpha_2-1)}\,.$$

For $b = 1$ and $\int_0^1 k(t)\,dt \le 1$ we have

$$|\ln u(x)| \le |\ln[C(\alpha_1)x^2]^{1/(\alpha_1-1)}| = \frac{1}{\alpha_1 - 1}|\ln[C(\alpha_1)x^2]|\,.$$

Hence

$$\begin{aligned} \varrho_{b,2}(T_1 u, T_2 v) \le \left(1 - \frac{\alpha_1}{\alpha_2}\right) & \frac{\left[\frac{\alpha_1-1}{\alpha_1}k(1)\right]^{\alpha_1/(\alpha_1-1)\alpha_2}}{\alpha_1 - 1} \\ & \times \sup_{0<x\le 1}\{x^{2(\alpha_2-\alpha_1)/(\alpha_1-1)\alpha_2(\alpha_2-1)}|\ln[C(\alpha_1)x^2]|\}\,. \end{aligned}$$

So we have proved

Theorem 3.2. *Let $u \in \Omega_1$ be a solution of the equation $u^{\alpha_1} = k * u$ where $\alpha_1 > 1$ is given and the kernel k satisfies* (1.1) *and $\int_0^1 k(t)\,dt \le 1$. If $v \in \Omega_1$ is a solution of $u^{\alpha_2} = k * u$ where $\alpha_1 < \alpha_2$ then*

$$\sup_{0\le x\le 1}|u(x) - v(x)| \le C(\alpha_1, \alpha_2, k)(\alpha_2 - \alpha_1)$$

with

$$C(\alpha_1,\alpha_2,k) = \frac{k'(0)e^{\beta}\left[\frac{\alpha_1-1}{\alpha_1}k(1)\right]^{\alpha_1/(\alpha_1-1)\alpha_2}}{\alpha_2 k'(0)-k'(a)} \times \sup_{0<x\le 1}\{x^{2(\alpha_2-\alpha_1)/(\alpha_1-1)\alpha_2(\alpha_2-1)}|\ln[C(\alpha_1)x^2]|\}.$$

We observe that theorems analogous to 3.1 and 3.2 can be proved for kernels k satisfying the conditions of Section 2.

4. Equations with nonhomogeneous linear part. Making use of the results contained in Sections 1–2 we investigate the solvability of the equation (0.2) with nonhomogeneous linear part as well as the problem of stability of solutions with respect to the perturbations of f. The section is divided into two parts A and B, according to the assumptions concerning the kernel k.

A. Consider equation (0.2) where $f\in\Gamma_0$ is a nonnegative, nondecreasing function satisfying $f''(x)\ge 0$ for $x\ge 0$ and $f(x)=O(x^{2\alpha/(\alpha-1)})$, $x\to 0$. The kernel k satisfies the conditions of Section 1.

LEMMA 4.1. *Any solution $u\in\Gamma_0$ of* (0.2) *is a nondecreasing function of class C^2 and*

$$C(\alpha)x^{2/(\alpha-1)} \le u(x) \le \left[\frac{\alpha-1}{\alpha}\int_0^x k(t)\,dt + (f(x))^{(\alpha-1)/\alpha}\right]^{1/(\alpha-1)}. \tag{4.1}$$

for $x>0$, where $C(\alpha)$ is defined by (1.2).

Put $(T_f u)(x) = (\int_0^x k(x-t)u(t)\,dt + f(x))^{1/\alpha}$ and denote by Ω_b'' the set $\{u\in C[0,b] : u \text{ satisfies (4.1)}\}$. Let $F(x)=C(\alpha)x^{2/(\alpha-1)}$.

LEMMA 4.2. *The operator T_f maps Ω_b'' into itself.*

The proofs of these two lemmas are similar to the proofs of the corresponding lemmas of Section 1, cf. also [9, 10].

We equip Ω_b'' with the same metric as in Ω_b (it is important that its weight is defined by the left-hand sides of (1.2) and (4.1), which are identical). Ω_b'' is a complete metric space. We have

$$|(T_f u_2)(x) - (T_f u_1)(x)| \le \frac{1}{\alpha}F^{1-\alpha}(x)|[k*(u_2-u_1)](x)|. \tag{4.2}$$

The right-hand side of (4.2) is independent of f and is of the same form as in Section 1 in similar estimates of $|T_0u_2 - T_0u_1|$ corresponding to the case $f=0$. Therefore T_f, just as T in Section 1, is a contraction and we have

THEOREM 4.1. *The nonlinear equation* (0.2) *of convolution type has a unique solution* u^* *in* Γ_0 (*and in* Ω_b'', $b \le \infty$). *It may be found by the method of successive approximations.*

Now we consider the problem of the dependence of solutions to equation (0.2) on f (see [9], Lemma 6).

THEOREM 4.2. *The following inequality holds true*:

$$\left(1 - \frac{k'(a)}{\alpha k'(0)}\right) \varrho_b(u_1, u_2) \le \frac{2(\alpha+1)}{k'(0)(\alpha-1)^2} \sup_{0<x\le b} \frac{|f_1(x) - f_2(x)|}{e^{\beta x} x^{2\alpha/(\alpha-1)}}, \tag{4.3}$$

where $u_j(x)$, $j = 1, 2$, *are the solutions of the equations* $u^\alpha = k * u + f_j$ *respectively.*

P r o o f. Making use of the Lagrange Theorem and (4.1) we have

$$\begin{aligned}|u_1(x) - u_2(x)| &\le \frac{[C(\alpha)]^{1-\alpha}}{\alpha x^2}\{|[k*(u_1 - u_2)](x)| + |f_1(x) - f_2(x)|\} \\ &\le \frac{[C(\alpha)]^{1-\alpha}}{\alpha x^2}\Big\{\varrho_b(u_1,u_2)\int_0^x k(x-t)e^{\beta t} t^{2/(\alpha-1)}\,dt + |f_1(x) - f_2(x)|\Big\}.\end{aligned}$$

Since

$$\begin{aligned}\int_0^x k(x-t)e^{\beta t}t^{2/(\alpha-1)}\,dt &= \int_0^x k(\tau)e^{-\beta\tau}e^{\beta x}(x-\tau)^{2/(\alpha-1)}\,d\tau \\ &\le k'(a)e^{\beta x}\int_0^x \tau(x-\tau)^{2/(\alpha-1)}\,d\tau \\ &= k'(a)e^{\beta x}\frac{(\alpha-1)^2}{2\alpha(\alpha+1)}x^2 x^{2/(\alpha-1)},\end{aligned}$$

the preceding inequality gives

$$\varrho_b(u_1,u_2) \le \frac{[C(\alpha)]^{1-\alpha}}{\alpha}\left\{\frac{k'(a)(\alpha-1)^2}{2\alpha(\alpha+1)}\varrho_b(u_1,u_2) + \sup_{0<x\le b}\frac{|f_1(x)-f_2(x)|}{e^{\beta x}x^{2\alpha/(\alpha-1)}}\right\}.$$

To get (4.3) we only have to note that

$$\frac{[C(\alpha)]^{1-\alpha}}{\alpha}\frac{k'(a)(\alpha-1)^2}{2\alpha(\alpha+1)} = \frac{k'(a)}{k'(0)}\frac{1}{\alpha}.$$

R e m a r k 4.1. Note that in contrast to [10] the results of Part A have been proved without assuming $f(x)x^{1/(1-\alpha)}$ to be nondecreasing and convex for $x > 0$ and, moreover, the proof of Theorem 4.2 does not require $\alpha = 2$.

B. Consider now equation (0.2) and assume that f is a nonnegative continuous function and k satisfies the conditions of Section 2.

From (4.2) it is easy to see that (0.2) has a unique solution in Ω'_b for any $b \le \infty$. Without repeating arguments similar to those used in the proof of Theorem 4.2 we content ourselves only with the formulation of the next theorem.

THEOREM 4.3. *Let* $f_j(x) \ge 0$ *and let* $u_j(x)$ *be a solution of* $u^\alpha = k*u+f_j$, $j = 1,2$, *satisfying* $u_j(x) \ge \gamma x^{(\nu+1)/(\alpha-1)}$, *where* γ *is defined by* (2.3). *Then*

$$\left(1 - \frac{p+\varepsilon}{\alpha p}\right) \varrho_b(u_1, u_2) \le \frac{1}{\alpha}\gamma^{1-\alpha} \sup_{0<x\le b} \frac{|f_1(x) - f_2(x)|}{e^{\beta x} x^{(\nu+1)\alpha/(\alpha-1)}} \,.$$

R e m a r k 4.2. The results of Section 3 concerning the dependence of solutions on k and α can be generalized to the case of equation (0.2).

5. Case $0 < \alpha < 1$

A. It has been shown in Section 1 that the nonlinear equation (0.1) may have for $\alpha > 1$ a nontrivial solution, therefore the theory of this equation must be essentially different from that of the linear case ($\alpha = 1$). It will be shown that for $\alpha < 1$ equation (0.1) has only the trivial solution, as in the linear case. We assume that k and f are nonnegative continuous functions on $[0,\infty)$. Let

$$N_j(y) = \max_{0\le x\le y} u_j(x), \qquad L(y) = \max_{0\le x\le y} f(x)\,,$$

$$D_y = \max_j \Big[N_i(y) \int_0^y k(t)\,dt + L(y)\Big]^{(1-\alpha)/\alpha}.$$

LEMMA 5.1. *If equation* (0.2) *has*, *for* $0 < \alpha < 1$, *a solution in* Γ, *then it is unique.*

P r o o f (some arguments of the theory of Volterra operators will be used). Let $u_j(x) = (T_f u_j)(x)$, $j = 1,2$, be two solutions of (0.2). The equation implies $u_1(0) = u_2(0) = f^{1/\alpha}(0)$. We shall prove that $u_1(x) = u_2(x)$ for small x. Assuming $x \in [0,b]$, $b < 1$, we have by the Lagrange Theorem

$$\begin{aligned}&|u_1(x) - u_2(x)| \\ &\le \frac{1}{\alpha}|[k*(u_1-u_2)](x)|(\max[(k*u_1+f)(x),(k*u_2+f)(x)])^{(1-\alpha)/\alpha} \\ &\le \frac{1}{\alpha}D_b|[k*(u_1-u_2)](x)| \le \frac{1}{\alpha}D_1\|u_1-u_2\|_{C[0,b]} \int_0^b k(t)\,dt\end{aligned}$$

and hence

$$\|u_1 - u_2\|_{C[0,b]} \le \left(\frac{1}{\alpha}D_1 \int_0^b k(t)\,dt\right)\|u_1 - u_2\|_{C[0,b]}\,.$$

Since D_1 and α are constants independent of b, we have $(1/\alpha)D_1\int_0^b k(t)\,dt < 1$ for small b, and hence $u_1(x) = u_2(x)$ for $0 \le x \le b$. Let $\bar{b} = \sup\{b : u_1(x) = u_2(x)$ for $0 \le x \le b\}$. Clearly $b \le \bar{b} \le \infty$. If $\bar{b} = \infty$ our lemma is proved. If $\bar{b} < \infty$, then u_1 and u_2 differ on $(\bar{b}, \bar{b}+\varepsilon)$ for any $\varepsilon > 0$. We show this to be impossible. Estimating as before $u_1(x) - u_2(x)$ for $x \in [\bar{b}, \bar{b}+\varepsilon]$ we obtain

$$\|u_1 - u_2\|_{C[\bar{b},\bar{b}+\varepsilon]} \le \left(\frac{1}{\alpha} D_{\bar{b}+1} \int\limits_0^{\varepsilon} k(t)\,dt\right) \|u_1 - u_2\|_{C[\bar{b},\bar{b}+\varepsilon]}\,,$$

which implies $u_1(x) = u_2(x)$ for $x \in [\bar{b}, \bar{b}+\varepsilon]$ for sufficiently small ε. This completes the proof.

COROLLARY. *If* $0 < \alpha < 1$ *then equation* (0.1) *has in* Γ *only a trivial solution.*

Note that for solutions belonging to $C[0,\infty)$ and bounded at infinity Lemma 5.1 was proved in [6].

B. If α, $0 < \alpha < 1$, takes some particular values, we may ask for solutions that are not necessarily nonnegative. To find them we make use of the method of monotonic operators due to Browder–Minty (cf. [4], [13]). We recall some basic facts of the theory.

Let E be a real Banach space and E^* its dual with norms $\|\cdot\|$ and $\|\cdot\|_*$ respectively. Let $\langle y, x\rangle$ denote the value of $y \in E^*$ at $x \in E$. Let $u, v \in E$.

DEFINITION 5.1. An operator $A : E \to E^*$ is said to be

monotonic if $\langle Au - Av, u - v\rangle \ge 0$,
strictly monotonic if $\langle Au - Av, u - v\rangle > 0$ for $u \ne v$,
strongly monotonic if $\langle Au - Av, u - v\rangle \ge m\|u - v\|^2$,
coercive if $\langle Au, u\rangle \ge \gamma(\|u\|)\|u\|$,

where $m > 0$, γ is a real function defined on $\mathbb{R}_+$ such that $\gamma(t) \to \infty$ as $t \to \infty$.

THEOREM (Browder's Principle [8, p. 326]). *Let E be a reflexive Banach space and let $A : E \to E^*$ be a continuous monotonic* (*resp. strictly monotonic*) *operator satisfying* $\langle Au, u\rangle \ge 0$ *for* $\|u\| = R_0$. *Then the equation* $Au = 0$ *has a solution* (*resp. a unique solution*) *in the ball* $\|u\| \le R_0$.

Note that for linear operators the conditions of monotonicity, strict monotonicity and strong monotonicity reduce to positivity, strict positivity and positive definiteness respectively.

Let $1 < p \le 2$ and $k \in L_1(\mathbb{R}) \cap L_{p/2(p-1)}(\mathbb{R})$, $\mathbb{R} = (-\infty, \infty)$. By Young's Theorem the convolution $(Hu)(x) = \int_{-\infty}^{\infty} k(x-t)u(t)\,dt$ defines a bounded operator from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$, $p^{-1} + q^{-1} = 1$.

Let $\widehat{u}$ denote the Fourier transform of u:

$$\widehat{u}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(t) e^{-ixt}\, dt\,,$$

and let $(\,\cdot\,,\,\cdot\,)$ be the inner product in $L_2(\mathbb{R})$, which coincides with $\langle\,\cdot\,,\,\cdot\,\rangle$ if E is a Hilbert space.

LEMMA 5.2. *The convolution operator* $H : L_p(\mathbb{R}) \to L_q(\mathbb{R})$, $1 \le p \le 2$, *is positive* (*resp. strictly positive*) *if and only if*

$$\operatorname{Re} \widehat{k}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k(t) \cos(xt)\, dt \ge 0, \qquad 0 \le x < \infty \tag{5.1}$$

(*resp.* $\operatorname{Re} \widehat{k}(x) > 0$). *If* $k \in L_{q/2}(\mathbb{R})$ *is an odd function then* $\langle Hu, u\rangle = 0$.

Proof. *Sufficiency.* We distinguish two cases.

(a) $p = 2$. Using well known relations (cf. formulas (2.18) and (2.29) in [14]) and $\overline{u(x)} = u(x)$ we obtain

$$\begin{aligned}(Hu, u) = (\widehat{Hu}, \overline{\widehat{u}}) = (\widehat{k}\widehat{u}, \overline{\widehat{u}}) &= \int_{-\infty}^{\infty} \widehat{k}(x) |\widehat{u}(x)|^2\, dx \\ &= \int_{-\infty}^{\infty} \operatorname{Re} \widehat{k}(x) |\widehat{u}(x)|^2\, dx + i \int_{-\infty}^{\infty} \operatorname{Im} \widehat{k}(x) |\widehat{u}(x)|^2\, dx\,.\end{aligned} \tag{5.2}$$

It is easy to see that $\operatorname{Re} \widehat{k}(x)$ is even and $\operatorname{Im} \widehat{k}(x)$ odd. Then from (5.2) and from the fact that $|\widehat{u}(x)|^2$ is even we obtain

$$(Hu, u) = \int_{-\infty}^{\infty} \operatorname{Re} \widehat{k}(x) |\widehat{u}(x)|^2\, dx\,, \tag{5.3}$$

and hence $(Hu, u) \ge 0$ owing to (5.1).

(b) $1 \le p < 2$. The density of $L_2(\mathbb{R}) \cap L_p(\mathbb{R})$ in $L_p(\mathbb{R})$, the continuity of $\langle Hu, u\rangle$ and (a) imply $\langle Hu, u\rangle \ge 0$ for all $u \in L_p(\mathbb{R})$.

Let now $k \in L_{q/2}(\mathbb{R})$ be an odd function. It has odd approximations $k_\varepsilon \in C_0^\infty$ such that $\|k - k_\varepsilon\|_{q/2} \to 0$ as $\varepsilon \to 0$. Let $H_\varepsilon u = k_\varepsilon * u$. We have

$$\|H - H_\varepsilon\|_{p\to q} \le \|k - k_\varepsilon\|_{q/2} \to 0 \qquad \text{as } \ \varepsilon \to 0\,. \tag{5.4}$$

k_ε is odd, therefore $\operatorname{Re} \widehat{k}_\varepsilon(x) = 0$ and (5.3) implies $\langle H_\varepsilon u, u\rangle = 0$. By (5.4), $\langle Hu, u\rangle = 0$ follows.

Necessity. Let $p = 2$ and $\langle Hu, u\rangle \geq 0$. If $\operatorname{Re}\widehat{k}(x_0) < 0$ for some $x_0 \geq 0$, then $\operatorname{Re}\widehat{k}(x) < 0$ in a neighbourhood $|x - x_0| < \varepsilon$ of x_0. Choose $\widehat{u}(x) = 1$ for $|x - x_0| < \varepsilon$ and $\widehat{u}(x) = 0$ for $|x - x_0| > \varepsilon$. $\widehat{u}$ (being in $L_2(\mathbb{R})$) is the Fourier transform of some $u \in L_2(\mathbb{R})$ for which

$$(Hu, u) = \int\limits_{|x-x_0|<\varepsilon} \operatorname{Re}\widehat{k}(x)\,dx < 0\,,$$

and that contradicts our assumption.

For $p \neq 2$ we can use the density argument as in (b).

COROLLARY 5.1. *If $p = 2$ then the operator H is not positive definite.*

Proof. If $(Hu, u) \geq m\|u\|_2^2$, $m > 0$, then by (5.3) and Parseval's identity we obtain

$$(Hu, u) - m\|u\|_2^2 = \int\limits_{-\infty}^{\infty} (\operatorname{Re}\widehat{k}(x) - m)|\widehat{u}(x)|^2\,dx \geq 0$$

for any $u \in L_2(\mathbb{R})$. This is obviously impossible owing to $\operatorname{Re}\widehat{k}(x) \to 0$ as $x \to \infty$, which implies $\operatorname{Re}\widehat{k}(x) - m < 0$ for large x.

COROLLARY 5.2. *If $p = 2$ and $\widehat{k} \in L_1(\mathbb{R})$ then the operator H is not coercive.*

Proof. It suffices to prove the existence of a sequence $\{u_n\}$ in $L_2(\mathbb{R})$ such that

$$\lim_{n\to\infty} \|u_n\|_2 = \infty, \qquad \lim_{n\to\infty} (Hu_n, u_n)/\|u_n\|_2 \neq \infty\,.$$

Put $\widehat{u}_n(x) = 1$ for $0 < x < n$ and $\widehat{u}_n(x) = 0$ elsewhere. It is clear that $u_n \in L_2(R)$, $\|u_n\|_2 = \|\widehat{u}_n\|_2 = \sqrt{n}$, $|(Hu_n, u_n)| \leq \|\widehat{k}\|_1$. Therefore

$$\lim_{n\to\infty} (Hu_n, u_n)/\|u_n\|_2 = 0\,,$$

which completes the proof.

COROLLARY 5.3. *If $1 < p \leq 2$, $k \in L_1(\mathbb{R}_+) \cap L_{q/2}(\mathbb{R}_+)$ and $\operatorname{Re}\widehat{k}(x) \geq 0$ (resp. $\operatorname{Re}\widehat{k}(x) > 0$) for $x \geq 0$, then the convolution operator $(Ku)(x) = \int_0^x k(x - t)u(t)\,dt$ is continuous and positive (resp. strictly positive) from $L_p(\mathbb{R}_+)$ to $L_q(\mathbb{R}_+)$.*

THEOREM 5.1. *Let $p = 2n/(2n - 1)$, $n = 1, 2, \ldots$, and $k \in L_1(\mathbb{R}_+) \cap L_n(\mathbb{R}_+)$. If $\operatorname{Re}\widehat{k}(x) \leq 0$ for $x \geq 0$ then for any $f \in L_{2n}(\mathbb{R}_+)$ and $\alpha = p - 1$ equation (0.2) has a unique solution $u^* \in L_p(\mathbb{R}_+)$ for which*

$$\|u^*\|_p \leq \|f\|_{2n}^{1/(p-1)}\,. \tag{5.5}$$

Proof. We rewrite (0.2) in the form $Au = 0$ with $Au = u^{p-1} - k * u - f$ and then apply Browder's Principle. The operator $A : L_p(\mathbb{R}_+) \to L_{2n}(\mathbb{R}_+)$

is continuous. By Corollary 5.3 we have, for any $u, v \in L_p(R_+)$, $u \neq v$,

$$\langle Au - Av, u - v\rangle = \langle u^{p-1} - v^{p-1}, u - v\rangle - \langle k * (u - v), u - v\rangle > 0\,,$$

hence A is strictly monotonic. Now

$$\begin{aligned}\langle Au, u\rangle &= \|u\|_p^p - \langle k * u, u\rangle - \langle f, u\rangle \\ &\geq \|u\|_p(\|u\|_p^{p-1} - \|f\|_{2n}) = 0 \quad \text{if } \|u\|_p = \|f\|_{2n}^{1/(p-1)} \equiv R_0\,,\end{aligned}$$

and the assertion follows from Browder's Principle.

The inequality (5.5) implies

COROLLARY 5.4. *Under the assumption of Theorem* 5.1 *the equation* $u^{p-1} = k * u$ *has in* $L_p(\mathbb{R}_+)$ *only a trivial solution.*

Note finally that the last corollary holds true for all $p \in (1, 2]$ provided $u(x) \geq 0$.

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