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ON CERTAIN UNIVALENCE CRITERIA

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1. Let $D = \{z : |z| < 1\}$ and let S_f denote the Schwarzian derivative

$$S_f = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f')z)}\right)^2 .$$

Epstein (see e.g. [4]) gave the following

THEOREM E. *Let f be meromorphic and g analytic in D. If both functions are locally univalent in D and if*

$$\left|\frac{1}{2}(1-|z|^2)^2(S_f(z)-S_g(z)) + (1-|z|^2)\overline{z}\frac{g''(z)}{g'(z)}\right| \le 1$$

for $z \in D$, then f is univalent in D.

Ch. Pommerenke [4] has given another proof of this theorem omitting an additional technical assumption introduced by Epstein.

We want to transfer this result to the exterior of the unit circle. We start with meromorphic functions in D. Note that we cannot simply apply the transformation $w \to 1/w$ to Theorem E, since g''/g' does not transform correctly.

2. THEOREM 1. *Let f and g be meromorphic functions in D of the form*

$$f(z) = \frac{a}{z} + a_0 + a_1 z + \ldots , \qquad g(z) = \frac{b}{z} + b_0 + b_1 z + \ldots$$

If both functions are locally univalent in D and if

$$\left|\left(\frac{1}{|z|^2} - 1\right)\left(2 + \frac{zg''(z)}{g'(z)}\right) + \frac{1}{2}(1-|z|^2)^2(S_f(z)-S_g(z))\right| \le 1, \tag{2.1}$$

for $z \in D$, then f is univalent in D.

Proof. We may assume $a = b = 1$, $a_0 = 0$. Let

$$v(z) = \sqrt{g'(z)/f'(z)} = 1 + \beta z^2 + \ldots , \tag{2.2}$$

$$u(z) = f(z)v(z) = \frac{1}{z} + c_1 z + \ldots \tag{2.3}$$

For $t \in I = [0, \infty)$ we consider (see e.g. [1], p. 38) the function

$$f(z,t) = \left(\frac{u(ze^{-t}) + (e^{-t} - e^{-3t})zu'(ze^{-t})}{v(ze^{-t}) + (e^{-t} - e^{-3t})zv'(ze^{-t})}\right)^{-1}, \quad z \in D. \tag{2.4}$$

For each fixed $t \in I$ the function $f(z,t)$ is meromorphic in D. From (2.2) and (2.3) it follows that there exist constants $r_0 > 0$ and K_0 such that

$$|f(z,t)| \le K_0 e^t \quad \text{for } |z| < r_0,\ t \in I. \tag{2.5}$$

From (2.2) and (2.3) we also conclude that

$$f(z,t) = e^t z + O(z^2) \quad \text{as } z \to 0\,. \tag{2.6}$$

Write

$$f'(z,t) = \frac{\partial f(z,t)}{\partial z}, \quad \dot{f}(z,t) = \frac{\partial f(z,t)}{\partial t}.$$

After some calculation we obtain from (2.4)

$$\begin{aligned} w &= \frac{\dot{f}(z,t) - zf'(z,t)}{\dot{f}(z,t) + zf'(z,t)} \\ &= -\frac{2ae^{3t}(u'v - uv') + az^2e^{2t}(u''v - uv'') + a^2z^2a^{2t}(u''v' - u'v'')}{u'v - uv'} \end{aligned} \tag{2.7}$$

where $a = e^{-t} - e^{-3t}$ and where $u, v, \ldots$ are evaluated at ze^{-t}.

From (2.3) together with (2.2) we obtain

$$\begin{aligned} u'v - uv' &= f'v^2, \\ u''v - uv'' &= f''v^2 + 2f'v'v = f'v^2g''/g', \\ u''v' - u'v'' &= f''v'v - f'v''v + 2f'(v')^2 = \tfrac{1}{2}f'v^2(S_f - S_g). \end{aligned}$$

Thus, it follows from (2.7) that, for $z \in D$,

$$\begin{aligned} w = &-(e^{2t} - 1)\left(2 + \frac{ze^{-t}g''(ze^{-t})}{g'(ze^{-t})}\right) \\ &- \frac{1}{2}(e^{2t} - 1)^2 z^2 e^{-4t}(S_f(ze^{-t}) - S_g(ze^{-t})). \end{aligned} \tag{2.8}$$

The right-hand side of (2.8) is 0 for $t = 0$ and is analytic in $\overline{D} = \{z : |z| \le 1\}$ if $t > 0$. Then, putting $ze^{-t} = \zeta$, $e^{-t} = |\zeta|$ and replacing ζ by z we deduce from (2.8) and (2.1) that

$$\left|\frac{\dot{f}(z,t) - zf'(z,t)}{\dot{f}(z,t) + zf'(z,t)}\right| \le 1,$$

so

$$\dot{f}(z,t) = zf'(z,t)p(z,t), \quad \operatorname{Re} p(z,t) > 0 \quad \text{for } z \in D,\ t \in I.$$

Thus, from (2.5) and (2.6) it follows that $f(z,t)$, $t \in I$, is a Löwner chain ([4], Theorem 6.2) and that $f(z,t)$ is univalent in D.

In particular, we conclude from (2.3) and (2.4) that

$$f(z,0) = 1/f(z) = v(z)/u(z)$$

is univalent in D.

Theorem 1, on substituting $F(z) = f(1/z)$, $G(z) = g(1/z)$, $z \in E = \{z : |z| > 1\}$, gives

THEOREM 2. *Let F and G be meromorphic and locally univalent in E. If*

$$\left|\frac{1}{2}(|z|^2-1)^2\frac{z}{\bar{z}}(S_F(z)-S_G(z))-(|z|^2-1)\frac{zG''(z)}{G'(z)}\right| \le 1, \quad z \in E, \tag{2.9}$$

then F is univalent in E.

If $G = F$ then (2.9) gives

$$(|z|^2-1)|zG''(z)/G'(z)| \le 1, \quad z \in E, \tag{2.10}$$

which is the known Becker univalence criterion [2].

If $G = z$ then (2.9) implies

$$(|z|^2-1)^2|S_F(z)| \le 2, \quad z \in E, \tag{2.11}$$

which is a univalence criterion of Nehari type [3].

EXAMPLE. We will now show that the functions

$$F(z) = \left(\frac{z}{z-1}\right)^{\sqrt{2}}, \quad G(z) = \frac{2z^2}{2z-1} \tag{2.12}$$

satisfy the assumptions of the univalence criterion (2.9) but G does not satisfy (2.10). Indeed, if $1 < x < 2$ then

$$(x^2-1)\frac{xG''(x)}{G'(x)} = \frac{x^2-1}{(x-1)(2x-1)} = \frac{x+1}{2x-1} > 1.$$

On the other hand,

$$\begin{aligned}
&\left|\frac{1}{2}(|z|^2-1)^2\frac{z}{\bar{z}}(S_F-S_G)-(|z|^2-1)\frac{zG''}{G'}\right| \\
&\quad = \frac{1}{2}\frac{|z|^2-1}{|z-1|}\left|\frac{|z|^2-1}{|z|^2(z-1)}-\frac{2}{2z-1}\right| = \frac{1}{2}\frac{(|z|^2-1)||z|^2-2z+1|}{|z|^2|z-1|^2|2z-1|} \\
&\quad \le \frac{1}{2}\frac{|z|^2-1}{|z|^2|2z-1|}\left(1+\frac{||z|^2-z|}{|z-1|^2}\right) \le \frac{1}{2}\frac{3r^2-1}{r^2(2r-1)} < 1.
\end{aligned}$$

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