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ON A PROBLEM OF FELL AND DORAN

BY

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Let X be a real or complex topological vector space. Denote by $L(X)$ the algebra of all continuous endomorphisms of X. Let A be an algebra over the same field of scalars as X. A *topological vector space representation* (shortly: a *t.v.s.-representation*) of A on X is a homomorphism T of A into $L(X)$. Denote by T_a the operator in $L(X)$ which is the value of T at an element a in A. A t.v.s.-representation T of A on X is said to be *irreducible* if every element x in X, $x \neq 0$, is cyclic for T, i.e. the orbit $\mathcal{O}(T;x) = \{T_a x \in X : a \in A\}$ is dense in X. In other words, T is irreducible if there is no closed proper subspace $X_0 \subset X$ (i.e. $\{0\} \neq X_0 \neq X$) which is invariant for all operators T_a, $a \in A$. For given T denote by $T^{(k)}$ the t.v.s.-representation of A on X^k—the k-fold direct sum of X—given by

$$T_a^{(k)}(x_1, \ldots, x_k) = (T_a x_1, \ldots, T_a x_k), \quad a \in A.$$

A t.v.s.-representation T of A on X is said to be *totally irreducible* if each vector $(x_1, \ldots, x_k)$ in X^k with linearly independent coordinates is cyclic for $T^{(k)}$ for $k = 1, 2, \ldots$

Thus T is totally irreducible if and only if for each positive integer k and k-tuple $(x_1, \ldots, x_k)$ of linearly independent elements of X the multiple orbit

$$\mathcal{O}(T; x_1, \ldots, x_k) = \{(T_a x_1, \ldots, T_a x_k) \in X^k : a \in A\} \tag{1}$$

is dense in X^k endowed with the cartesian product topology. Let T and S be two t.v.s.-representations of A respectively on X and Y. Let R be a linear densely defined operator from X into Y with domain D_R. It is said to be *intertwining between T and S*, or *(T,S)-intertwining*, if $T_a D_R \subset D_R$ for all a in A and

$$RT_a x = S_a R x \tag{2}$$

for all x in D_R and all a in A. We do not assume the continuity of R. If R is continuous, we can extend it by continuity onto the whole of X, and since relations (2) will be satisfied for all x in X, by continuity of the involved operators, we can assume in this case $D_R = X$.

In ([1], problem II, p. 321) Fell and Doran ask whether an irreducible locally convex representation of an algebra A on X (i.e. a t.v.s.-representation on X which is a locally convex space), such that the only continuous (T,T)-intertwining operators are scalar multiples of the identity, is necessarily totally irreducible. In this paper we give necessary and sufficient conditions in order that the answer to this question be in the affirmative. Our main result reads as follows.

THEOREM 1. *Let X be a real or complex topological vector space and let T be an irreducible t.v.s.-representation of an algebra A on X for which the only continuous (T,T)-intertwining operators are scalar multiples of the identity. Then T is totally irreducible if and only if all closed $(T,T^{(k)})$-intertwining operators are continuous for all positive integers k.*

In the above we say, as usual, that a densely defined operator R from X into Y is closed if its graph

$$\Gamma_R = \{(x,Rx) \in X \times Y : x \in D_R\}$$

is a closed subset in $X \times Y$.

This theorem is a corollary to the following more technical result. To formulate it we need the following

DEFINITION. Let R be a densely defined operator from X into X^n, for some positive integer n. We say that R is a *scalar operator* if there are scalars $\lambda_1,\ldots,\lambda_n$ such that

$$Rx = (\lambda_1 x,\ldots,\lambda_n x)\,, \qquad x \in D_R\,. \tag{3}$$

THEOREM 2. *Let X be a real or complex topological vector space and let A be an algebra over the same field of scalars as X. Let T be an irreducible t.v.s.-representation of A on X. Then T is totally irreducible if and only if all closed $(T,T^{(k)})$-intertwining operators are scalar for all positive integers k.*

Proof. Suppose that there is a positive integer k such that some closed $(T,T^{(k)})$-intertwining operator R is non-scalar. Its graph Γ_R is a proper closed subspace of X^{k+1}. Since R is non-scalar, there is an element $(z_0,\ldots,z_k)$ in Γ_R, with $z_0 \in D_R$, $(z_1,\ldots,z_k) = Rz_0$ and with

$$\dim \operatorname{span}\{z_0,\ldots,z_k\} = n+1 > 1\,. \tag{4}$$

Relation (2) with $S = T^{(k)}$ implies

$$RT_a z_0 = T_a^{(k)}(z_1,\ldots,z_k) = (T_a z_1,\ldots,T_a z_k)\,,$$

and since $T_a z_0 \in D_R$ for all a, it follows that

$$\{(T_a z_0,\ldots,T_a z_k) \in X^{k+1} : a \in A\} \subset \Gamma_R\,.$$

By (4) we can choose $i_1, \dots, i_n$ so that the elements $z_0, z_{i_1}, \dots, z_{i_n}$ are linearly independent and put

$$Q = \{(T_a z_0, T_a z_{i_1}, \dots, T_a z_{i_n}) \in X^{n+1} : a \in A\}.$$

We claim that the closure $\overline{Q}$ of Q in X^{n+1} is a proper closed subspace in this space. In fact, by irreducibility of T we have $Q \neq (0)$. If $\overline{Q}$ coincides with X^{n+1} we can choose there an element of the form $(0, y_{i_1}, \dots, y_{i_n})$ with $y_{i_n} \neq 0$. Since, by (4), $z_j = \alpha_0 z_0 + \sum_{s=1}^{n} \alpha_s^j z_{i_s}$ for $j \neq 0, i_1, \dots, i_n$, $0 < j \leq k$, the element $(0, y_1, \dots, y_k)$ is in Γ_R, where

$$y_j = \sum_{s=1}^{n} \alpha_s^j y_{i_s}, \qquad 0 < j \leq k,\ j \neq i_1, \dots, i_n.$$

But this is absurd, since $y_{i_n} \neq 0$ and Γ_R is the graph of an operator. Thus $\overline{Q} \neq X^{n+1}$. Since the closure of the orbit $\mathcal{O}(T; z_0, z_{i_1}, \dots, z_{i_n})$ is contained in $\overline{Q}$, the vector $(z_0, z_{i_1}, \dots, z_{i_n})$ is non-cyclic for $T^{(n)}$. Since the coordinates $z_0, z_{i_1}, \dots, z_{i_n}$ are linearly independent the representation T is not totally irreducible.

Suppose now that each closed $(T, T^{(k)})$-intertwining operator is scalar for $k = 1, 2, \dots$ We have to show that for each positive integer k and each k-tuple $(x_1, \dots, x_k)$ of linearly independent elements of X the orbit (1) is dense in X^k. We shall prove this by induction on k. For $k = 1$ it follows immediately from the definition of an irreducible t.v.s.-representation. Suppose now that for every $k \leq n$ and every k-tuple $x_1, \dots, x_k$ of linearly independent elements of X the orbit (1) is dense in X^k. We have to show that for any linearly independent elements $x_1, \dots, x_{n+1}$ in X the orbit $\mathcal{O}(T; x_1, \dots, x_{n+1})$ is dense in X^{n+1}. Let $\overline{\mathcal{O}} = \overline{\mathcal{O}}(T; x_1, \dots, x_{n+1})$ be the closure of this orbit in X^{n+1}.

In the first step we shall show that $\overline{\mathcal{O}} = X^{n+1}$ if and only if there is $(z_1, \dots, z_{n+1}) \in \overline{\mathcal{O}}$ such that $z_{i_0} \neq 0$ for some i_0, $1 \leq i_0 \leq n+1$, and $z_i = 0$ for $i \neq i_0$. One of the implications is trivial. So suppose that we have such an element $(z_1, \dots, z_{n+1})$ in $\overline{\mathcal{O}}$. First we show that for every y in X the element $(y_1, \dots, y_{n+1})$ is in $\overline{\mathcal{O}}$, where $y_{i_0} = y$ and $y_i = 0$ for $i \neq i_0$.

The relation $T_a^{(n+1)} \mathcal{O}(T; x_1, \dots, x_{n+1}) \subset \mathcal{O}(T; x_1, \dots, x_{n+1})$ implies $T_a^{(n+1)} \overline{\mathcal{O}} \subset \overline{\mathcal{O}}$, for all a in A, by the continuity of the operators $T_a^{(n+1)}$. This implies $\mathcal{O}(T; z_1, \dots, z_{n+1}) \subset \overline{\mathcal{O}}$ and since by the irreducibility of T we have $(y_1, \dots, y_{n+1}) \subset \overline{\mathcal{O}}(T; z_1, \dots, z_{n+1})$, we obtain $(y_1, \dots, y_{n+1}) \in \overline{\mathcal{O}}$. To conclude the first step it is sufficient to show that for any j, $1 \leq j \leq n+1$, there is an element $(y_1, \dots, y_{n+1})$ in $\overline{\mathcal{O}}$ with $y_i = 0$ for $i \neq j$ and $y_j = y$, an arbitrarily given element in X. Summing such elements over j we can obtain an arbitrary element in X^{n+1}. For $j = i_0$ we are already done, so suppose $j \neq i_0$. Denote by $\Phi(X)$ a basis of neighbourhoods of the origin in X. Fix y in X and U in $\Phi(X)$ and take a V in $\Phi(X)$ with $V + V \subset U$. By the inductive assumption the orbit $\mathcal{O}(T; u_1, \dots, u_n)$ is dense in X^n, where

$u_1, \ldots, u_n$ are the elements $x_1, \ldots, x_{i_0-1}, x_{i_0+1}, \ldots, x_{n+1}$. Thus there is an $a(V)$ in A such that

$$T_{a(V)}x_j \in y + V\,, \quad T_{a(V)}x_i \in V \quad \text{for } i_0 \neq i \neq j\,. \tag{5}$$

Since for $j = i_0$ we are already done, we can find an element $b(V)$ in A such that

$$T_{b(V)}x_{i_0} \in -T_{a(V)}x_{i_0} + V\,, \quad T_{b(V)}x_i \in V \quad \text{for } i \neq i_0\,,\ 1 \leq i \leq n+1\,. \tag{6}$$

Adding coordinatewise (5) and (6) and using the first relation in (6) we obtain

$$\begin{gathered} T_{a(V)+b(V)}x_i \in V + V \subset U \quad \text{for } i_0 \neq i \neq j\,, \\ T_{a(V)+b(V)}x_{i_0} \in V \subset V + V \subset U\,, \\ T_{a(V)+b(V)}x_j \in y + V + V \subset y + U\,. \end{gathered}$$

Since U was chosen arbitrarily in $\Phi(X)$, we have $(y_1, \ldots, y_n) \in \overline{\mathcal{O}}$, where $y_i = 0$ for $i \neq j$, $1 \leq i \leq n+1$, and $y_j = y$. The proof of the first step is complete.

Consider now two mutually excluding cases:

(a_1) No non-zero element $(z_1, \ldots, z_{n+1})$ in $\overline{\mathcal{O}}$ has a zero coordinate.

(a_2) There is a non-zero $(n+1)$-tuple $(z_1, \ldots, z_{n+1})$ in $\overline{\mathcal{O}}$ with some coordinate z_{i_0}, $1 \leq i_0 \leq n+1$, equal to zero.

In the case (a_1) the linear space $\overline{\mathcal{O}}$ is the graph of the closed operator R from X to X^n given by

$$Rz_1 = (z_2, \ldots, z_{n+1})\,, \quad (z_1, \ldots, z_{n+1}) \in \overline{\mathcal{O}}\,. \tag{7}$$

It is a well defined operator on its domain D_R, which is the projection of $\overline{\mathcal{O}}$ onto the first coordinate space. Thus D_R is a dense subset of X, since it contains the orbit $\mathcal{O}(T; x_1)$. We claim that R is a $(T, T^{(n)})$-intertwining operator. To see this, we use the inclusions

$$T_a^{(n+1)}\overline{\mathcal{O}} \subset \overline{\mathcal{O}}\,, \tag{8}$$

for all a in A, obtained in the first step of our proof. They imply $T_aD_R \subset D_R$ for all a in A. Moreover, by (7) and (8) we have

$$RT_az_1 = (T_az_2, \ldots, T_az_{n+1}) = T_a^{(n)}(z_2, \ldots, z_{n+1}) = T_a^{(n)}Rz_1$$

for all a in A and all z_1 in D_R. Our claim is proved.

Since R is a closed $(T, T^{(n)})$-intertwining operator, it is scalar. So, for example, $z_2 = \lambda z_1$ for some scalar λ and this holds for all elements $(z_1, z_2, \ldots, z_{n+1})$ in $\overline{\mathcal{O}}$. In particular, we have $T_ax_2 = \lambda T_ax_1$, or $T_a(x_2 - \lambda x_1) = 0$ for all a in A. This contradicts the irreducibility of T, since $x_2 - \lambda x_1$ is a non-zero element in X. Thus we must have the case (a_2). In this case there are elements in $\overline{\mathcal{O}}$ having some, but not all, coordinates equal to zero. Among them there must exist elements with the maximal number, say s, of zero coordinates in the sense that if some $(z_1, \ldots, z_{n+1})$ in $\overline{\mathcal{O}}$ has more than s coordinates equal to zero, then all of them are zeroes. Fix such an element $(z_1^0, \ldots, z_{n+1}^0)$. After a suitable renumbering of $x_1, \ldots, x_{n+1}$ we can assume $z_1^0 = \ldots = z_s^0 = 0$, and all other coordinates $z_{s+1}^0, \ldots, z_{n+1}^0$ are different from zero. We have $s \le n$, if $s = n$ we are done by the first step of the proof. So assume $1 \le s < n$ and consider two cases

(b_1) $\dim \operatorname{span}(z_{s+1}^0, \ldots, z_{n+1}^0) = k > 1$, and
(b_2) $\dim \operatorname{span}(z_{s+1}^0, \ldots, z_{n+1}^0) = 1$.

In the case (b_1) we fix k linearly independent elements $z_{i_1}^0, \ldots, z_{i_k}^0$, $s < i_l \le n+1$. We have

$$z_i^0 = \sum_{l=1}^{k} \alpha_l^i z_{i_l}^0 \quad \text{for } s < i \le n+1\,, \tag{9}$$

where α_l^i are suitable scalars. We have $k \le n$, and so by the inductive assumption the orbit $\mathcal{O}(T; z_{i_1}^0, \ldots, z_{i_k}^0)$ is dense in X^k. Thus for an arbitrary $y \ne 0$ in X and an arbitrary neighbourhood U in $\Phi(X)$ we can choose a V in $\Phi(X)$ so that $V + V \subset U$ and choose an element $a(U)$ in A so that

$$T_{a(V)} z_{i_l}^0 \in V \quad \text{for } 1 \le l < k\,, \qquad T_{a(V)} z_{i_k}^0 \in y + V\,. \tag{10}$$

By the continuity of $T_{a(V)}$ we can find another neighbourhood V_1 in $\Phi(X)$ so that

$$\begin{gathered} T_{a(V)} V_1 \subset V\,, \qquad V_1 \subset V\,, \\ \sum_{l=1}^{k-1} \alpha_l^i V_1 \subset V \quad \text{for all } i \text{ with } s < i \le n+1\,, \end{gathered} \tag{11}$$

where α_l^i are the coefficients in formula (9). Because $(z_1^0, \ldots, z_{n+1}^0)$ is in $\overline{\mathcal{O}}$, there is a $b(V_1)$ in A such that

$$T_{b(V_1)} x_i \in z_i^0 + V_1\,, \qquad i = 1, \ldots, n+1\,.$$

Thus by (10) and (11) we obtain

$$\begin{aligned} T_{a(V)b(V_1)} x_i &\in T_{a(V)}(z_i^0 + V_1) \\ &\subset \begin{cases} V + V \subset U & \text{for } i = 1, \ldots, s, i_1, \ldots, i_{k-1}, \\ y + V + V \subset y + U & \text{for } i = i_k, \\ \alpha_k^i y + V + V \subset \alpha_k^i y + U & \text{for } i \ne 1, \ldots, s, i_1, \ldots, i_k. \end{cases} \end{aligned} \tag{12}$$

Since U was arbitrarily chosen, we see from (12) that $(y_1, \ldots, y_{n+1})$ is in $\overline{\mathcal{O}}$, where $y_1 = \cdots = y_s = y_{i_1} = \ldots = y_{i_{k-1}} = 0$ and $y_{i_k} = y$ while $y_i = \alpha_k^i y$ for $i \ne 1, \ldots, s, i_1, \ldots, i_k$. Since $k \ge 2$ and $y \ne 0$ we have obtained a non-zero element in $\overline{\mathcal{O}}$ which has more than s zero coordinates. This contradicts the definition of s, and thus the case (b_1) cannot occur.

We then have the case (b_2) and the element $(z_1^0, \ldots, z_{n+1}^0)$ has the coordinates $z_1^0 = \ldots = z_s^0 = 0$, $z_i^0 = \lambda_i z_{n+1}^0$, $s < i \le n$, $z_{n+1}^0 \neq 0$, and all the scalars λ_i are different from zero. Define $y_i = x_i$ for $1 \le i \le s$ and $i = n+1$, and $y_i = x_i - \lambda_i x_{n+1}$, $s < i \le n$. Consider the map M of X^{n+1} onto itself given by

$$M(u_1, \ldots, u_{n+1}) = (u_1, \ldots, u_s, u_{s+1} - \lambda_{s+1} u_{n+1}, \ldots, u_n - \lambda_n u_{n+1}, u_{n+1}).$$

It is a one-to-one continuous map with a continuous inverse. It extends the map of $\mathcal{O}(T; x_1, \ldots, x_{n+1})$ onto $\mathcal{O}(T; y_1, \ldots, y_{n+1})$ given by $(T_a x_1, \ldots, T_a x_{n+1}) \to (T_a y_1, \ldots, T_a y_{n+1})$. Hence M maps $\overline{\mathcal{O}}$ onto the closure $\overline{\mathcal{O}}_1$ of $\mathcal{O}_1 = \mathcal{O}(T; y_1, \ldots, y_{n+1})$ in X^{n+1}. Thus $\overline{\mathcal{O}}_1$ contains $M(z_1^0, \ldots, z_{n+1}^0) = (0, \ldots, 0, z_{n+1}^0)$. Since $z_{n+1}^0 \neq 0$ we obtain by the first step of this proof the equality $\overline{\mathcal{O}}_1 = X^{n+1}$. Consequently, $\overline{\mathcal{O}} = M^{-1}\overline{\mathcal{O}}_1 = X^{n+1}$ and the conclusion follows.

Proof of Theorem 1. If T is totally irreducible, then by Theorem 2 each closed $(T, T^{(k)})$-intertwining operator is scalar, and so continuous. On the other hand, suppose that each $(T, T^{(k)})$-intertwining operator R is continuous. Writing it in the form $Rx = (R_1 x, \ldots, R_k x)$ we see that all operators R_i are in $L(X)$. We also have

$$RT_a x = (R_1 T_a x, \ldots, R_k T_a x) = T_a^{(k)} Rx = (T_a R_1 x, \ldots, T_a R_k x),$$

and this holds for all a in A and all x in X. Thus the R_i are continuous (T, T)-intertwining operators, so by the assumption of Theorem 1 there are scalars λ_i with $R_i x = \lambda_i x$, $x \in X$. This implies that R is scalar and so, by Theorem 2, T is totally irreducible. The conclusion follows.

We cannot solve the problem of Fell and Doran formulated above even if X is a Banach space. However, if the answer is in the negative, as some specialists believe, our Theorem 1 and also Theorem 2 offer an additional condition under which the answer is affirmative. On the other hand, if the answer is affirmative, our Theorem 1 offers a way of attacking this problem, since it reduces it to the more specific investigation of closed $(T, T^{(k)})$-intertwining operators.

In [4] we solved the problem in the affirmative for representations of algebras on completely metrizable topological vector spaces (F-spaces) under the additional assumption that the representations T in question are algebraically irreducible, i.e. all orbits $\mathcal{O}(T; x)$, $x \neq 0$, coincide with the whole of X (in the conclusion we do not have algebraic total irreducibility but merely total irreducibility). This result can also be obtained as a corollary to Theorem 1 of the present paper. The results presented here, though formulated for general topological vector spaces, make unrestricted sense only for locally convex spaces. This is caused by the fact that for a topological vector space X the algebra $L(X)$ can be very poor, and there are so-called (infinite-dimensional) rigid spaces X (see [2] and [3]) for which $L(X)$ contains only scalar multiples of the identity operator, so that there are no t.v.s.-representations on X which are irreducible.

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