COLLOQUIUM MATHEMATICUM

VOL. LXII 1991 FASC. I

ON POMMERENKE'S INEQUALITY FOR THE EIGENVALUES OF FIXED POINTS

BY

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§1. Introduction. One of the main results of the paper is the following. We investigate the existence of solutions of the equation

$$\lambda h(\omega) = h(\rho\omega), \quad |\lambda| > 1, \ \rho > 1, \tag{1.1}$$

in the class of mappings which are K-quasi-conformal in an open semidisc D centred at zero. The image of the diameter of the semidisc may be an arbitrary boundary subset of $h(D)$. Such a situation arises in iteration theory of polynomial and polynomial-like mappings. In those cases h maps the exterior of the unit disc (or equivalently a half plane) to the basin of attraction of infinity and ρ is the degree of the mapping. We shall prove in particular that

$$|\ln\lambda|^2/\ln|\lambda| \le 2K\ln\rho \tag{1.2}$$

and determine all cases when equality occurs in (1.2).

Actually, (1.2) implies a generalization of the following theorem by Ch. Pommerenke [7]:

THEOREM 1 [7]. *Let $a \neq \infty$ be a repulsive fixed point of a rational function f ($\deg f \ge 2$). For $i = 1, \ldots, p$, let Ω_i be the distinct simply connected invariant components of $\overline{\mathbb{C}} \setminus J$ ($J = J(f)$ denotes the Julia set for f [4], [5], [6]), let h_i map conformally the unit disc onto Ω_i and let ω_{ik}, $|\omega_{ik}| = 1$, be distinct fixed points of the conjugate mappings $\varphi_i = h_i^{-1} \circ f \circ h_i$ with*

$$h_i(\omega_{ik}) = a, \quad k = 1, \ldots, l_i. \tag{1.3}$$

Then

$$\sum_{i=1}^{p}\sum_{k=1}^{l_i} \frac{1}{\ln\varphi_i'(\omega_{ik})} \le \frac{2\ln|f'(a)|}{|\ln f'(a)|^2} \le \frac{2}{\ln|f'(a)|}. \tag{1.4}$$

Note that φ_i is a finite Blaschke product and ω_{ik} is a repulsive fixed point of φ_i. Equality (1.3) is to be understood to mean that the angular limit $\lim_{\omega \underset{\sphericalangle}{\to} \omega_{ik}} h_i(\omega) = a$ exists [7].

In the present paper we shall prove (1.4) in a more general situation. Our method is related to the extremal lengths method [1]. It allows us to investigate when equality is achieved in (1.4).

Notations:

$D(r)=\{\omega:|\omega|<r\,,\ \mathrm{Im}\,\omega>0\}$,
$\Pi=\{\omega:\mathrm{Im}\,\omega>0\}$, $B(r)=\{z:|z|<r\}$,
$C(r_1,r_2)=\{z:r_1<|z|<r_2\}$,
$z_0A=\{z:\exists u\in A\,,\ z=z_0u\}$ $(z_0\in\mathbb{C},\ A\subset\mathbb{C})$

For example:

$$\Pi=\bigcup_{k=0}^{\infty}\rho^kD(r)\,,\qquad \rho>1\,,\ r>0\,.$$

§2. Results. Let $f:z\mapsto\lambda z$, $\varphi_\rho:\omega\mapsto\rho\omega$, $|\lambda|>1$, $\rho>1$. Suppose there exist domains Ω, U and a mapping h_0 such that

(1) $0\in\partial\Omega$, $\Omega\subset\lambda\Omega\subset\mathbb{C}$, $0\in\partial U$, $U\subset\rho U\subset\mathbb{C}$, $\bigcup_{n=0}^{\infty}\rho^nU=\Pi$;

(2) $h_0:\rho U\to\lambda\Omega$ is a K-quasi-conformal homeomorphism [5] which conjugates $f_\lambda|_\Omega$ and $\varphi_\rho|_U$:

$$\lambda h_0(\omega)=h_0(\rho\omega)\,,\qquad \omega\in U\,. \tag{2.1}$$

We shall prove the following basic

THEOREM 2. (a) *We have*

$$|\ln\lambda|^2/\ln|\lambda|\le 2\alpha^*K\ln\rho\,, \tag{2.2}$$

where

$$\alpha^*=\lim_{\delta\to0}\frac{1}{2\pi\ln(r/\delta)}\iint\limits_{\Omega\cap C(\delta,r)}|z|^{-2}\,dx\,dy\,,\qquad z=x+iy\,,\ r>0\,;$$

(b) *equality is achieved in* (2.2) *if and only if*

$$h_0(\omega)=\xi\omega^\eta\overline{\omega}^\kappa\,,\qquad \xi,\eta,\kappa\in\mathbb{C}\,,\ \kappa=t\eta\,,\ t\in[0,1)\,;$$

under this condition the boundary of the domain

$$\Omega^*=\bigcup_{n=0}^{\infty}\lambda^n\cdot\Omega$$

is limited by either rays (*if* $\lambda>0$), *or logarithmical spirals.*

Remark 1. The number α^* equals the density of the domain Ω at 0 in the logarithmic metric $|dz|/|z|$.

We now formulate a generalization of Theorem 1. Let $f:A\to\mathbb{C}$ be a map conformal in a neighbourhood A of 0, and let $f(0)=0$, $f'(0)=\lambda$,

$|\lambda| > 1$. Suppose there exist finitely many pairwise disjoint domains Ω_i and mappings h_i, $i = 1, \ldots, p$, such that

(1′) $0 \in \partial\Omega_i$, $\Omega_i \subset f(\Omega_i) \subset A$;

(2′) for every i there exist $\varepsilon_i > 0$, $K_i \geq 1$ and $\rho_i > 1$ for which $h_i : D(\rho_i\varepsilon_i) \to f(\Omega_i)$ is K_i-quasi-conformal with

$$f(h_i(\omega)) = h_i(\rho_i\omega), \quad \omega \in D(\varepsilon_i).$$

Theorem 3. (a) *We have*

$$\sum_{i=1}^{p} \frac{1}{K_i} \cdot \frac{1}{\ln \rho_i} \leq \frac{2\underline{\alpha} \ln |\lambda|}{|\ln \lambda|^2}, \tag{2.3}$$

where

$$\underline{\alpha} = \underline{\lim}_{\delta \to 0} \frac{1}{2\pi \ln(r/\delta)} \iint_{\Omega \cap C(\delta, r)} |z|^{-2}\, dx\, dy,$$

the lower density of $\Omega = \bigcup_{i=1}^{p} \Omega_i$ *at* 0 *in the logarithmic metric.*

(b) *If equality holds in* (2.3), *then every* h_i *extends continuously to a closed semi-neighbourhood* $\overline{D(\varepsilon_i)}$ *of* $\omega = 0$ *and transforms the boundary interval to an analytic arc with end at* $z = 0$.

Remark 2. Theorem 1 follows from Theorem 3 if Schröder's theorem [9] is applied. Then φ_i is locally (in neighbourhood of ω_{ik}) conjugate to its derivative $\omega \mapsto \varphi_i'(\omega_{ik})\omega$. Besides, $K_i = 1$.

Corollary. *Equality is achieved in the left inequality of* (1.4) *if and only if the Julia set of* f *is either a circle or a segment and* a *is any fixed point of* f.

The proofs are given in §§ 3, 4. Hyperbolic sets are introduced in § 5. The results of §§ 3–5 are applied in § 6 for estimation of eigenvalues of polynomials and polynomial-like mappings periodic points. The paper is ended by some comments and open problems.

§ 3. Proof of Theorem 2

3.1. The mapping h_0 may be extended to a mapping h of the half-plane Π with the property (2.1). The extension is given by

$$h(\rho^n\omega) = \lambda^n h_0(\omega), \quad n = 0, 1, \ldots;\ \omega \in U.$$

We get a K-quasi-conformal homeomorphism $h : \Pi \to \Omega^*$, where

$$\Omega^* = h(\Pi) = \bigcup_{k=0}^{\infty} \lambda^k \cdot \Omega, \quad \lambda h(\omega) = h(\rho\omega), \quad \omega \in \Pi.$$

3.2. For every ray

$$\alpha_\varphi = \{\omega \in \Pi \mid \arg \omega = \varphi\}, \quad 0 < \varphi < \pi,$$

we have

$$\lim_{\omega\to 0} h(\omega) = 0\,, \qquad \lim_{\omega\to\infty} h(\omega) = \infty\,,$$

if $\omega \in \alpha_\varphi$.

3.3. Now we fix the boundary circle S_r of a ball $B(r)$ and consider the curve $\beta_{\varphi_0} = h(\alpha_{\varphi_0})$ with some $\varphi_0 \in (0,\pi)$. This curve is in Ω^* and joins 0 and ∞. There exists an arc $S \subset S_r \cap \Omega^*$ with ends on $\partial\Omega^*$ through which β_{φ_0} leaves the ball $B(r)$. Then any β_φ crosses S. Set

$$l = h^{-1}(S)\,.$$

Every ray α_φ crosses l, $0 < \varphi < \pi$.

3.4. We now introduce two families of curves $\widetilde{\Gamma}$ and Γ. Consider first the family of all intervals joining points $\omega \in l$ and ω/ρ; then on every ray α_φ, $0 < \varphi < \pi$, we choose exactly one such interval $\widetilde{\gamma} = \widetilde{\gamma}_\varphi$, namely the one closest to zero. We get the family of intervals $\{\widetilde{\gamma}_\varphi\} = \widetilde{\Gamma}$. It fills in some set $R \subset \Pi$.

The family Γ is the family of images $\gamma = h(\widetilde{\gamma})$, $\widetilde{\gamma} \in \widetilde{\Gamma}$; every curve $\gamma \in \Gamma$ joins a point $z \in S$ and z/λ. The family Γ fills in the set $h(R) \subset \Omega^*$.

Now introduce the logarithmic metric in $\mathbb{C} \setminus \{0\}$:

$$\sigma(z) = 1/|z|\,, \qquad z \neq 0\,,$$

and the induced metric in Π:

$$\widetilde{\sigma}(\omega) = \frac{\sigma(z)}{|(h^{-1})'_z| - |(h^{-1})'_{\bar z}|}\bigg|_{z=h(\omega)}\,.$$

Define (see [1])

$$L = \inf_{\gamma\in\Gamma} \int_\gamma \sigma(z)\,|dz|\,, \qquad A = \iint_{h(R)} \sigma^2(z)\,dx\,dy\,,$$

$$\widetilde{L} = \inf_{\widetilde{\gamma}\in\widetilde{\Gamma}} \int_{\widetilde{\gamma}} \widetilde{\sigma}(\omega)\,|d\omega|\,, \qquad \widetilde{A} = \iint_R \widetilde{\sigma}^2(\omega)\,du\,dv\,,$$

($z = x + iy$, $\omega = u + iv$) and, finally,

$$M = m(\sigma,\Gamma) = A/L^2\,, \qquad \widetilde{M} = m(\widetilde{\sigma},\widetilde{\Gamma}) = \widetilde{A}/\widetilde{L}^2\,.$$

3.5. We prove that

$$M \geq \widetilde{M}/K \tag{3.1}$$

(this is a general fact, see [1]). Let $\gamma = h(\widetilde{\gamma})$, $\widetilde{\gamma} \in \widetilde{\Gamma}$. Then

$$\int_{\widetilde{\gamma}} \widetilde{\sigma}(\omega)\,|d\omega| \geq \int_\gamma \sigma(z)\,|dz|\,, \tag{3.2}$$

$$\iint_R \widetilde{\sigma}^2(\omega)\,du\,dv \le K \iint_{h(R)} \sigma^2(z)\,dx\,dy\,, \tag{3.3}$$

and (3.1) follows.

3.6. We estimate $\widetilde{M}$ from below. For every $\widetilde{\gamma}_\varphi \in \widetilde{\Gamma}$ we have

$$\widetilde{L}^2 \le \Big(\int_{\widetilde{\gamma}_\varphi} \widetilde{\sigma}\,|d\omega| \Big)^2 \le \int_{\widetilde{\gamma}_\varphi} \widetilde{\sigma}^2 \cdot |\omega|\,|d\omega| \cdot \int_{\widetilde{\gamma}_\varphi} \left|\frac{d\omega}{\omega}\right|\,. \tag{3.4}$$

But

$$\int_{\widetilde{\gamma}_\varphi} \left|\frac{d\omega}{\omega}\right| = \ln\rho\,,$$

therefore

$$\pi\widetilde{L}^2 \le \ln\rho \cdot \int_0^\pi d\varphi \int_{\widetilde{\gamma}_\varphi} \widetilde{\sigma}^2(re^{i\varphi})r\,dr = \ln\rho \cdot \iint_R \widetilde{\sigma}^2\,du\,dv = \ln\rho\cdot\widetilde{A}\,.$$

Thus,

$$\widetilde{M} \ge \pi/\ln\rho\,. \tag{3.5}$$

3.7. Now we estimate M from above. Firstly,

$$\int_\gamma \sigma(z)\,|dz| \ge \left| \int_\gamma \frac{dz}{z} \right| = |\ln\lambda|\,, \qquad \gamma \in \Gamma\,. \tag{3.6}$$

Secondly, consider

$$A = \iint_{h(R)} \frac{dx\,dy}{|z|^2}\,.$$

Let $z \in h(R)$ and suppose z, $z_1 = \lambda z$ and $z_2 = z/\lambda$ are not endpoints of any curve $\gamma \in \Gamma$. It follows from the definition of the family $\widetilde{\Gamma}$ that $z_1, z_2 \notin h(R)$.

Denote the area of a set $V \subset \mathbb{C} \setminus \{0\}$ in the logarithmic metric by

$$I(V) = \iint_V \frac{dx\,dy}{|z|^2}\,.$$

For example, $A = I(h(R))$. Obviously,

$$I(V) = I(\lambda V)\,. \tag{3.7}$$

Now, if z belongs to $h(R)$, but not to the annulus

$$C = C(r/|\lambda|, r) = \{z : r/|\lambda| < |z| < r\}\,,$$

then we transform z into C by the mapping $z \mapsto \lambda^k z$ with some $k = \pm1, \pm2, \dots$ By the above,

$$A = I(h(R)) \le I(\Omega^* \cap C)\,.$$

We now show that

$$\frac{I(\Omega^* \cap C)}{2\pi \ln|\lambda|} = \alpha^* = \lim_{\delta \to 0} \frac{I(\Omega \cap C(\delta, r))}{I(C(\delta, r))}\,.$$

Indeed, this follows from (3.7):

$$\begin{aligned} I(\Omega^* \cap C) &= \lim_{k\to\infty} I(C \cap \lambda^k \Omega) = \lim_{k\to\infty} I(C(r|\lambda|^{-k-1}, r|\lambda|^{-k}) \cap \Omega) \\ &= \lim_{k\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} I(C(r|\lambda|^{-k-1}, r|\lambda|^{-k}) \cap \Omega) \\ &= \lim_{n\to\infty} \frac{1}{n} I(C(r|\lambda|^{-n}, r) \cap \Omega)\,, \end{aligned}$$

where the existence of each subsequent limit follows from the existence of the preceding one.

Hence the following limit exists:

$$\lim_{\delta\to 0} \frac{1}{\ln(r/\delta)} I(C(\delta, r) \cap \Omega) = \frac{I(\Omega^* \cap C)}{\ln|\lambda|} = 2\pi\alpha^*\,. \tag{3.8}$$

Thus, we have proved that

$$M \le \frac{2\pi\alpha^* \ln|\lambda|}{|\ln\lambda|^2}\,,$$

and, finally,

$$\frac{1}{K} \cdot \frac{\pi}{\ln\rho} \le \frac{1}{K} \widetilde{M} \le M \le \frac{2\pi\alpha^* \ln|\lambda|}{|\ln\lambda|^2}\,.$$

Part (a) of Theorem 2 is proved.

We proceed to prove (b). Suppose equality holds in (2.2). From (3.2) we obtain $L = \widetilde{L}$. Therefore we have equality in (3.5) and in Schwarz's inequality (3.4) (for almost every $\varphi \in (0, \pi)$). Hence

$$\widetilde{\sigma}(\omega) = \frac{\text{const}}{|\omega|} \tag{3.9}$$

almost everywhere on $\widetilde{\gamma}_\varphi$.

Now (3.2) may be rewritten as

$$\text{const} \cdot \ln\rho \ge \int_\gamma \sigma(z)\,|dz| \ge |\ln\lambda|\,.$$

From $L = \widetilde{L}$ it follows that

$$\text{const} \cdot \ln\rho = \int_\gamma \left|\frac{dz}{z}\right| = |\ln\lambda|$$

almost everywhere in φ. So, γ is a geodesic in the metric $|dz|/|z|$. Hence

$$h(\omega) = \xi \cdot \omega^\eta \overline{\omega}^\kappa\,.$$

The conditions on η and κ are verified by calculations. Theorem 2 is proved.

At the same time we have proved

LEMMA 1. *If a domain* Ω *is such that* $0 \in \partial\Omega$, $\Omega \subset \lambda\Omega$, $|\lambda| > 1$, *then*:

(a) *the limit*

$$\alpha = \lim_{\delta\to 0} \frac{I(\Omega \cap C(\delta, r))}{I(C(\delta, r))}, \qquad r > 0,$$

exists;

(b) α *is a conformal invariant*, *that is*, *for every mapping* ψ *conformal in a neighbourhood* V *of* 0 *and such that* $\psi(0) = 0$,

$$\alpha = \lim_{\delta\to 0} \frac{I(\psi(V \cap \Omega) \cap C(\delta, r))}{I(C(\delta, r))}.$$

§4. Proof of Theorem 3. By Schröder's theorem [9] (applied to the branch of f^{-1} with $f^{-1}(0) = 0$), there exists a conformal isomorphism $g : B(|\lambda|\tau) \to A_0$ from some ball $B(|\lambda|\tau)$ to a neighbourhood $A_0 \subset A$ of zero such that $g(\lambda z) = f(g(z))$, $z \in B(\tau)$.

Let $\Pi_t = \{\omega : \arg\omega \in (t, \pi - t)\}$. For $t \in (0, 1/2)$, the restriction of h_i to Π_t is continuous up to the point $\omega = 0$. Hence for every i and t there exists $\varepsilon = \varepsilon(i, t)$ such that for $U_{i,t} = D(\varepsilon) \cap \Pi_t$ we have

$$V_{i,t} = h_i(U_{i,t}) \subset A_0.$$

We may assume that $\varepsilon(i, t_1) < \varepsilon(i, t_2)$ if $0 < t_1 < t_2 < 1/2$. Set

$$U_i = \bigcup_t U_{i,t}, \qquad V_i = \bigcup_t V_{i,t} \subset A_0 \cap \Omega_i,$$

$$\widetilde{\Omega}_i = g^{-1}(V_i), \qquad V = \bigcup_i V_i \subset A_0 \cap \Omega, \qquad \widetilde{\Omega} = \bigcup_i \widetilde{\Omega}_i.$$

Then $\bigcup_{k=0}^{\infty} \rho_i^k U_i = \Pi$, $i = 1, \ldots, p$. We now apply Theorem 2 and Lemma 1 to get

$$\sum_{i=1}^{p} \frac{1}{K_i} \cdot \frac{1}{\ln \rho_i} \le \frac{2\alpha \ln|\lambda|}{|\ln\lambda|^2} \le \frac{2\underline{\alpha} \ln|\lambda|}{|\ln\lambda|^2},$$

where

$$\alpha = \lim_{\delta\to 0} \frac{I(V \cap C(\delta, r))}{I(C(\delta, r))} \le \varliminf_{\delta\to 0} \frac{I(\Omega \cap C(\delta, r))}{I(C(\delta, r))} = \underline{\alpha}.$$

Part (a) is thus proved; (b) follows from Theorem 2(b).

Proof of the Corollary. We apply Theorem 3(b) and the following theorem of Fatou [5]: if the Julia set J of a rational function contains an analytic arc, then J is a circle or a segment. Equality in (1.4) is checked up directly.

§5. Hyperbolic sets. Call a domain $\Omega \subset \mathbb{C}$ *hyperbolic* with (hyperbolicity) constant α, $0 < \alpha < 1$, if there exists $\varepsilon > 0$ such that for any ball $B_z(r)$ with centre at $z \in \partial\Omega$ and radius $r < \varepsilon$

$$\frac{l_2(B_z(r) \cap \Omega)}{l_2(B_z(r))} \le \alpha \tag{5.1}$$

(l_2 is the two-dimensional Lebesgue measure on $\mathbb{C}$).

EXAMPLE. Let Ω be the simply connected basin of attraction of an attracting fixed point $\xi \in \overline{\mathbb{C}}$ of a rational function f (more generally: Ω and f are the RB-domain and the mapping, introduced in [8]). Let $f : \partial\Omega \to \partial\Omega$ be an expanding mapping [7], that is, there exist $K > 1$, $n \in \mathbb{N}$ such that $|(f^n)'| > K$ on $\partial\Omega$. Then Ω satisfies (5.1) (see [7]).

Let $C_a(r_1, r_2) = \{z : r_1 < |z - a| < r_2\}$.

LEMMA 2. *If Ω is a hyperbolic domain with constant α, then for any $a \in \partial\Omega$ and any $r > 0$*

$$\varliminf_{\delta\to 0} \frac{1}{2\pi \ln(r/\delta)} I(C_a(\delta, r) \cap \Omega) \le \alpha\,.$$

Proof. Fix any $\alpha_1 > \alpha$ and choose $m \in (0, 1)$ so that

$$\alpha_1 = \frac{\alpha}{1 - m^2}\,.$$

Then for any $a \in \partial\Omega$ and $u < \varepsilon$

$$\frac{l_2(C_a(mu, u) \cap \Omega)}{l_2(C_a(mu, u))} \le \frac{l_2(B_a(u) \cap \Omega)}{(1 - m^2) l_2(B_a(u))} \le \alpha_1\,,$$

or

$$\int_{mu}^{u} l(\tau)\, d\tau \le \alpha_1 \int_{mu}^{u} 2\pi\tau\, d\tau\,, \qquad u \in (0, \varepsilon)\,.$$

Here $l(\tau)$ is the Euclidean length of that part of the circumference $|z - a| = \tau$ which lies in Ω. We substitute $\tau = ut$, $t \in (m, 1)$, divide the last inequality by u^3 and integrate over u from δ to r. We obtain

$$\int_m^1 dt \int_\delta^r \frac{l(ut)}{u^2}\, du \le 2\pi\alpha_1 \ln\frac{r}{\delta} \int_m^1 t\, dt\,,$$

or

$$\int_m^1 t\, dt \frac{1}{\ln\frac{rt}{\delta t}} \int_{\delta t}^{rt} \frac{l(\tau)}{\tau^2}\, dt \le 2\pi\alpha_1 \int_m^1 t\, dt\,. \tag{5.2}$$

Now define

$$\varliminf_{\delta\to 0} \frac{1}{\ln\frac{rt}{\delta t}} \int\limits_{\delta t}^{rt} \frac{l(\tau)}{\tau^2}\, d\tau \equiv A \le 2\pi\,.$$

A does not depend on t; from (5.2), $A \le 2\pi\alpha_1$, $\forall\alpha_1 > \alpha$. Thus, $A \le 2\pi\alpha$. Notice that

$$\int\limits_{\delta}^{r} \frac{l(\tau)}{\tau^2}\, d\tau = \iint\limits_{C_a(\delta,r)\cap\Omega} \frac{dx\,dy}{|z|^2} = I(C_a(\delta,r)\cap\Omega)\,.$$

§6. Applications. Let us write down the obtained results for polynomial-like mappings [8]. First, let P be a polynomial of degree $m \ge 2$ and suppose its Julia set $J(P)$ is connected. This is equivalent to the basin of attraction of infinity

$$D_\infty = \{z : P^n z \to \infty\,,\ n \to \infty\}\,, \qquad P^n = \underbrace{P\circ\ldots\circ P}_{n}\,,$$

being simply connected in the Riemann sphere $\overline{\mathbb{C}}$. There exists an analytic homeomorphism

$$H_0 : B(1) = \{z : |z| < 1\} \to D_\infty\,, \qquad H_0(0) = \infty\,.$$

The mapping H_0 transforms $P : D_\infty \to D_\infty$ into $P_0 : B(1) \to B(1)$, $P_0(\omega) = \omega^m$:

$$P \circ H_0 = H_0 \circ P_0\,.$$

Let $z_0 \in J(P)$ be a repulsive periodic point of P. Then z_0 can be reached by a curve from D_∞ and there exist a finite number r of radial directions in $B(1)$ on which $H_0(\omega) \to z_0$ ($|\omega| \to 1$) [2], [3].

Now consider a polynomial-like mapping $T : W \to W'$. This means that W, W' are simply connected domains, $\overline{W} \subset W'$ and $T : W \to W'$ is a proper holomorphic mapping of degree m, $m \ge 2$. The term "polynomial-like" is accounted for by Douady–Hubbard's theorem [2]: there exist a polynomial P of degree m and a quasi-conformal homeomorphism H_1 of some neighbourhood V of

$$F(P) = \{z : \sup_n |P^n z| < \infty\} = \mathbb{C} \setminus D_\infty$$

onto some neighbourhood U of

$$F(T) = \{z \in W : T^n z \in W\,,\ \forall n \in \mathbb{N}\}$$

such that $T \circ H_1(z) = H_1 \circ P(z)$ if $P(z) \in V$.

Denote the maximal dilation of the quasi-conformal mapping $H_1 : V \to U$ by K. Let $J(T) = \partial F(T)$.

Assume that the set $J(T)$ is connected and let $a \in J(T)$ be a repulsive periodic point of T with period n and eigenvalue

$$\lambda = (T^n)'(a).$$

Then we define $r = r(a)$ to be the (finite) number of radial directions in $B(1)$ on which $H_1 \circ H_0(\omega) \to a$. Theorem 3 and Lemma 2 yield

THEOREM 4.

(a) (1) $\dfrac{|\ln \lambda^l|^2}{\ln |\lambda^l|} \le \dfrac{2Kn \ln m^l}{r}$ *for some* $l \in \mathbb{N}$;

(b) *if every critical point of* T *is attracted by an attractive periodic cycle*, *then there exists* α, $0 < \alpha < 1$, *such that for any repulsive periodic point of* T *with eigenvalue* λ,

$$\frac{|\ln \lambda^l|^2}{\ln |\lambda^l|} \le \frac{2K\alpha n \ln m^l}{r} \quad \textit{for some } l \in \mathbb{N}.$$

Proof. (a) This follows from the fact that the eigenvalue ρ of any repulsive periodic point ω_0 with period N for the mapping $P_0 : \omega \mapsto \omega^m$ is $\rho = m^N$.

(b) is a consequence of the fact that the domain $\Omega = W' \setminus W$ is hyperbolic under the condition of (b) (see example in §5 and [7]).

§7. Comments and open problems. The inequality

$$\frac{1}{n} \ln |\lambda| \le 2 \ln m \tag{7.1}$$

for the eigenvalue λ of a periodic point of period n of a polynomial P ($\deg P = m$) with connected Julia set follows from Theorem 4. It may also be proved by the methods of entire function theory [3]. Rewrite it as

$$\chi_n \le 2\chi(P), \tag{7.2}$$

where $\chi_n = (1/n) \ln |\lambda|$ is the characteristic exponent of the periodic point and

$$\chi(P) = \int \ln |P'(z)|\, d\mu(z)$$

is the characteristic exponent of the dynamical system $P : J \to J$ related to the measure of maximal entropy or, equivalently,

$$\chi(P) = \lim_{k\to\infty} \overline{\chi}_k,$$

where $\overline{\chi}_k$ is the arithmetic mean of χ_k over all repulsive periodic points of period k.

(1) I was informed by the referee that the similar result was proved by Yoccoz [10] for polynomials.

QUESTION: does inequality (7.2) remain true for polynomials with disconnected Julia set? and for rational functions R ($\deg R \geq 2$)?

If $P(z) = z^m + c$, then (7.2) is true for every $c \in \mathbb{C}$.

ANOTHER PROBLEM: find the infimum x_* of x such that the inequality $\chi_n \leq x \ln m$ is valid for all periodic points of a given polynomial. We have proved that $x_* \leq 2$, and if $P : J \to J$ is expanding that $x_* < 2$ ($J(P)$ is connected). As shown in [3], either $x_* > 1$ or P is equivalent to z^m.

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Reçu par la Rédaction le 20.12.1989