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THE REIDEMEISTER ZETA FUNCTION AND THE COMPUTATION OF THE NIELSEN ZETA FUNCTION

BY

A. L. FEL′SHTYN (LENINGRAD)

§1. Introduction

1.1. *Preliminaries.* We assume everywhere X to be a connected compact polyhedron and $f : X \to X$ to be a continuous map. Let $p : \widetilde{X} \to X$ be the universal covering of X and $\widetilde{f} : \widetilde{X} \to \widetilde{X}$ a lifting of f, i.e. $p \circ \widetilde{f} = f \circ p$. Two liftings $\widetilde{f}$ and $\widetilde{f}'$ are called *conjugate* if there is $\gamma \in \varGamma \cong \pi_1(X)$ such that $\widetilde{f}' = \gamma \circ \widetilde{f} \circ \gamma^{-1}$. The subset $p(\mathrm{Fix}(\tilde{f})) \subset \mathrm{Fix}\, f$ is called the *fixed point class of f determined by the lifting class* $[\tilde{f}]$. A fixed point class is called *essential* if its index is nonzero. The number of lifiting classes of f (and hence the number of fixed point classes, empty or not) is called the *Reidemeister number* of f, denoted by $R(f)$. It is a positive integer or infinity. The number of essential fixed point classes is called the *Nielsen number* of f, denoted by $N(f)$. The Nielsen number is always finite. $R(f)$ and $N(f)$ are homotopy invariants.

We may define a few dynamical zeta functions in Nielsen fixed point theory (see [1, 5, 6, 12]). The *Reidemeister* and *Nielsen zeta functions* are defined as power series:

$$R_f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{R(f^n)}{n} z^n\right), \quad N_f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{N(f^n)}{n} z^n\right).$$

$R_f(z)$ and $N_f(z)$ are homotopy invariants. We study $R_f(z)$ in §3 and then compute $N_f(z)$ via $R_f(z)$ in §4.

Let G be a group and $\varphi : G \to G$ an endomorphism. Two elements $\alpha, \alpha' \in G$ are said to be *φ-conjugate* iff there exists $\gamma \in G$ such that $\alpha' = \gamma \cdot \alpha \cdot \varphi(\gamma^{-1})$. The number of φ-conjugacy classes is called the *Reidemeister number* of φ, denoted by $R(\varphi)$. We assume everywhere that $R(\varphi^n) < \infty$ for every $n > 0$ and consider the *Reidemeister zeta function* of φ,

$$R_\varphi(z) = \exp\left(\sum_{n=1}^{\infty} \frac{R(\varphi^n)}{n} z^n\right),$$

introduced in [5, 6]. We study $R_\varphi(z)$ in §2.

The results of this paper were partly announced in [6].

1.2. *Historical notes.* Nielsen developed his theory of fixed point classes and defined the number bearing his name in his study of surface homeomorphisms in 1927, using non-Euclidean geometry as a tool. Through the hands of Reidemeister and Wecken, it became a beautiful theory applicable to self-maps of polyhedra. Reidemeister gave a combinatorial treatment and considered the number bearing his name in 1936 [13]. It is interesting that the Lefschetz numbers

$$L(f) = \sum_{k=0}^{\dim X} (-1)^k \operatorname{tr}[f_{*k} : H_k(X,\mathbb{R}) \to H_k(X,\mathbb{R})]$$

appeared almost simultaneously [10] with the Nielsen numbers, but the Lefschetz zeta function

$$L_f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{L(f^n)}{n} z^n\right) = \prod_{k=0}^{\dim X} \det(E - f_{*k} \cdot z)^{(-1)^{k+1}}$$

was defined by A. Weil [17] in 1949 when he studied the fixed points of iterates of the Frobenius endomorphism. In the theory of discrete dynamical systems the Lefschetz zeta function was introduced by Smale in 1967 [15].

§ 2. The Reidemeister zeta function of a group endomorphism

Problem. For which groups and endomorphisms the Reidemeister zeta function is a rational function? Is $R_\varphi(z)$ an algebraic function?

When $R_\varphi(z)$ is a rational function the infinite sequence $\{R(\varphi^n)\}_{n=1}^{\infty}$ of Reidemeister numbers is determined by a finite set of complex numbers—the zeros and poles of $R_\varphi(z)$.

Lemma 1. *$R_\varphi(z)$ is a rational function if and only if there exists a finite set of complex numbers α_i and β_j such that $R(\varphi^n) = \sum_j \beta_j^n - \sum_i \alpha_i^n$ for every $n > 0$.*

Proof. Suppose $R_\varphi(z)$ is a rational function. Then

$$R_\varphi(z) = \prod_i (1 - \alpha_i z) / \prod_j (1 - \beta_j z),$$

where $\alpha_i, \beta_j \in \mathbb{C}$. Taking the logarithmic derivative of both sides and then using the geometric series expansion we see that $R(\varphi^n) = \sum_j \beta_j^n - \sum_i \alpha_i^n$. The converse is proved by a direct calculation. ∎

An endomorphism $\varphi : G \to G$ is said to be *eventually commutative* if there exists a natural number n such that the subgroup $\varphi^n(G)$ is commutative.

We are now ready to compare the Reidemeister zeta function of an endomorphism φ with the Reidemeister zeta function of $H_1(\varphi) : H_1(G) \to H_1(G)$, where $H_1 = H_1^{Gp}$ is the first integral homology functor from groups to abelian groups.

LEMMA 2. *If $\varphi : G \to G$ is eventually commutative, then*

$$R_\varphi(z) = R_{H_1(\varphi)}(z) = \exp\Big(\sum_{n=1}^{\infty} \frac{\operatorname{ord}\operatorname{Coker}(1 - H_1^n(\varphi))}{n} z^n\Big).$$

Proof. $R(\varphi^n) = R((H_1(\varphi))^n) = \operatorname{ord}\operatorname{Coker}(1 - H_1^n(\varphi))$ (see [7, 9]). ∎

THEOREM 1. *Suppose that $H_1(G)$ is torsion-free. Let φ be eventually commutative and assume that no eigenvalue of $H_1(\varphi)$ is a root of unity. Then $R_\varphi(z)$ is a rational function and equals*

$$R_\varphi(z) = \Big(\prod_{i=0}^{\operatorname{rg} H_1(G)} \det(E - \textstyle\bigwedge^i H_1(\varphi) \cdot \sigma z)^{(-1)^{i+1}}\Big)^{(-1)^r} \tag{1}$$

where $\sigma = (-1)^p$, p is the number of $\mu \in \operatorname{Spec} H_1(\varphi)$ such that $\mu < -1$, and r is the number of real $\lambda \in \operatorname{Spec} H_1(\varphi)$ such that $|\lambda| > 1$; $\bigwedge^i$ denotes the exterior power.

Proof. From the assumptions of the theorem it follows that $R(\varphi^n) = R(H_1^n(\varphi)) = \operatorname{ord}\operatorname{Coker}(1 - H_1^n(\varphi))$ for every $n > 0$.

Now we have

$$\operatorname{ord}\operatorname{Coker}(1 - H_1^n(\varphi)) = |\det(E - H_1^n(\varphi))| \neq 0\,.$$

Hence $R(\varphi^n) = (-1)^{r+pn}\det(E - H_1^n(\varphi))$. It is well known from linear algebra that $\det(E - H_1^n(\varphi)) = \sum_{i=0}^k (-1)^i \operatorname{tr} \bigwedge^i H_1^n(\varphi)$. Then we have the "trace formula" for the Reidemeister numbers:

$$R(\varphi^n) = (-1)^{r+pn} \sum_{i=0}^{k} (-1)^i \operatorname{tr} \textstyle\bigwedge^i H_1^n(\varphi)\,. \tag{2}$$

From (2) it follows that

$$\begin{aligned}
R_\varphi(z) &= \exp\Big(\sum_{n=1}^{\infty} \frac{R(\varphi^n)}{n} z^n\Big)\\
&= \exp\Big(\sum_{n=1}^{\infty} \frac{(-1)^r \cdot \sum_{i=0}^k (-1)^i \operatorname{tr} \bigwedge^i H_1^n(\varphi)}{n} (\sigma z)^n\Big)\\
&= \Big(\prod_{i=0}^{k}\Big(\exp\Big(\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{tr} \textstyle\bigwedge^i H_1^n(\varphi) \cdot (\sigma z)^n\Big)\Big)^{(-1)^i}\Big)^{(-1)^r}
\end{aligned}$$

$$= \Big(\prod_{i=0}^{k} \det(E - \bigwedge^i H_1(\varphi) \cdot \sigma z)^{(-1)^{i+1}} \Big)^{(-1)^r} . \ \blacksquare$$

COROLLARY 1. *Let the assumptions of Theorem* 1 *hold. Then the poles and zeros of the Reidemeister zeta function* $R_\varphi(z)$ *are complex numbers which are reciprocal to the eigenvalues of the matrices* $\bigwedge^i H_1(\varphi) \cdot \sigma$, $0 \le i \le \operatorname{rg} H_1(G)$.

PROPOSITION 1. *Let the assumptions of Theorem* 1 *hold. Then the functional equation for the Reidemeister zeta function* $R_\varphi(z)$ *is*

$$R_\varphi\left(\frac{1}{dz}\right) = (R_\varphi(z))^{(-1)^{\operatorname{rg} H_1(G)}} \cdot \varepsilon \,, \tag{3}$$

where $d = \det H_1(\varphi)$ *and* ε *is a complex number.*

Proof. Via the natural nonsingular pairing $(\bigwedge^i H_1(G)) \wedge (\bigwedge^{k-i} H_1(G)) \to \mathbb{C}$ the operators $\bigwedge^{k-i} H_1(\varphi)$ and $d(\bigwedge^i H_1(\varphi))^{-1}$ are adjoint to each other. Fix an eigenvalue λ of $\bigwedge^i H_1(\varphi)$. It contributes a term $(1-\lambda/(dz))^{(-1)^{i+1}}$ to $R_\varphi(1/(dz))$. Write this term as

$$\left(1 - \frac{dz}{\lambda}\right)^{(-1)^{i+1}} \cdot \left(\frac{-dz}{\lambda}\right)^{(-1)^i}$$

and note that d/λ is an eigenvalue of $\bigwedge^{k-i} H_1(\varphi)$. Now multiply over all λ. One finds that

$$\varepsilon = \Big(\prod_{i=1}^{\operatorname{rg} H_1(G)} \prod_{\lambda^{(i)} \in \operatorname{Spec} \bigwedge^i H_1(\varphi)} (1/\lambda^{(i)})^{(-1)^i} \Big)^{(-1)^r} .$$

The variable z disappears because

$$\sum_{i=0}^{k} (-1)^i \dim \bigwedge^i H_1(G) = \sum_{i=0}^{k} (-1)^i C_k^i = 0 \,. \ \blacksquare$$

THEOREM 2. *Suppose that* $\varphi : G \to G$ *is eventually commutative and* $H_1(G) = Z_p$ $(p > 1$ *prime*$)$. *Then* $R_\varphi(z)$ *is a rational function.*

Proof. For every $n > 0$, $R(\varphi^n) = \operatorname{ord} \operatorname{Coker}(1 - H_1^n(\varphi))$. Let $H_1(\varphi)(1) = d$. Then $(1 - H_1^n(\varphi))(Z_p) = (1 - d^n)Z_p$. So $\operatorname{Coker}(1 - H_1^n(\varphi)) = Z_p/(1-d^n)Z_p$, which is known to be the cyclic group of order $(1-d^n, p)$. If $p|d$ then $R(\varphi^n) = 1$ for every $n > 0$ and $R_\varphi(z) = 1/(1-z)$. If $(p,d) = 1$ then $d^{p-1} \equiv 1 \pmod p$ and the sequence $R(\varphi^n)$ is periodic with period k $(1 \le k \le p-1$ and $k|p-1)$. Thus $R(\varphi^n) = p$ if $k|n$ and $R(\varphi'') = 1$ otherwise.

Direct calculation shows that

$$R_\varphi(z) = \frac{(1-z^k)^{(1-p)/k}}{1-z} . \ \blacksquare$$

We will write $[\alpha]$ for the φ-conjugacy class of $\alpha \in G$.

LEMMA 3 [9]. *For any* $\alpha \in G$ *we have* $[\alpha] = [\varphi(\alpha)]$.

We say that $\varphi : G \to G$ is *nilpotent* if for some positive integer n, $\varphi^n : G \to G$ is the trivial homomorphism.

THEOREM 3. *If* φ *is nilpotent, then* $R_\varphi(z) = 1/(1-z)$.

Proof. For any $\alpha \in G$ we have $[\alpha] = [\varphi(\alpha)] = [\varphi^n(\alpha)] = [e]$, i.e. $R(\varphi) = 1$. The same is true for every $n > 0$. $\blacksquare$

2.1. *The Reidemeister zeta function and group extensions.* Suppose we are given a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G \\ \downarrow p & & \downarrow p \\ \overline{G} & \xrightarrow{\overline{\varphi}} & \overline{G} \end{array} \tag{4}$$

of groups and homomorphisms. In addition let the sequence

$$0 \longrightarrow H \longrightarrow G \longrightarrow \overline{G} \longrightarrow 0 \tag{5}$$

be exact. Then φ restricts to an endomorphism $\varphi|H : H \to H$.

DEFINITION 1. The short exact sequence (5) of groups is said to have a *normal splitting* if there is a section $\sigma : \overline{G} \to G$ of p such that $\operatorname{Im} \sigma = \sigma(\overline{G})$ is a normal subgroup of G. An endomorphism $\varphi : G \to G$ is said to *preserve* this normal splitting if φ induces a morphism of (5) with $\varphi(\sigma(\overline{G})) \subset \sigma(\overline{G})$.

In this section we study the relation between the Reidemeister zeta functions $R_\varphi(z)$, $R_{\overline{\varphi}}(z)$ and $R_{\varphi|H}(z)$.

THEOREM 4. *Let the sequence* (5) *have a normal splitting which is preserved by* $\varphi : G \to G$. *Suppose that* $R_{\overline{\varphi}}(z)$ *and* $R_{\varphi|H}(z)$ *are rational functions. Then so is* $R_\varphi(z)$.

Proof. From the assumptions of the theorem it follows that for every $n > 0$

$$R(\varphi^n) = R(\overline{\varphi}^n) \cdot R(\varphi^n|H) \qquad \text{(see [7])} .$$

Lemma 1 implies that there exist finite sets of complex numbers α_i, β_j and μ_i, ν_j such that

$$R(\overline{\varphi}^n) = \sum_j \beta_j^n - \sum_i \alpha_i^n, \quad R(\varphi^n|H) = \sum_j \nu_j^n - \sum_i \mu_i^n .$$

Then $R(\varphi^n) = (\sum_j \beta_j^n - \sum_i \alpha_i^n) \cdot (\sum_j \nu_j^n - \sum_i \mu_i^n)$. Now we multiply out and again use Lemma 1. ■

2.2. *Infinite product formula.* Let $\mu(d), d \in \mathbb{N}$, be the Möbius function, i.e.

$$\mu(d) = \begin{cases} 1 & \text{if } d = 1, \\ (-1)^k & \text{if } d = \prod_{i=1}^k p_i,\ p_i \text{ distinct primes}, \\ 0 & \text{if } p^2 | d \text{ for some prime } p. \end{cases}$$

We define the numbers $S(d)$, $d \in \mathbb{N}$, by

$$S(d) = \sum_{d_1|d} \mu(d_1) R(\varphi^{d/d_1}) .$$

THEOREM 5.

$$R_\varphi(z) = \prod_{d=1}^{\infty} \sqrt[d]{(1 - z^d)^{-S(d)}} . \tag{6}$$

P r o o f. Since $S(n) = \sum_{d|n} \mu(d) R(\varphi^{n/d})$, we have $R(\varphi^n) = \sum_{d|n} S(d)$ by the Möbius Inversion Theorem. Hence

$$\begin{aligned} R_\varphi(z) &= \exp\left(\sum_{n=1}^{\infty} \frac{R(\varphi^n)}{n} z^n \right) \\ &= \exp\left(\sum_{n=1}^{\infty} \frac{\sum_{d|n} S(d)}{n} z^n \right) = \exp\left(\sum_{d=1}^{\infty} \sum_{k=1}^{\infty} \frac{S(d)}{dk} z^{dk} \right) \\ &= \exp\left(\sum_{d=1}^{\infty} \frac{-S(d)}{d} \ln(1 - z^d) \right) = \prod_{d=1}^{\infty} \sqrt[d]{(1 - z^d)^{-S(d)}} . \ ■ \end{aligned}$$

§3. The Reidemeister zeta function of a continuous map. Let $f : X \to X$ be given, and let a specific lifting $\widetilde{f} : \widetilde{X} \to \widetilde{X}$ be chosen as reference. Then every lifting of f can be uniquely written as $\gamma \circ \widetilde{f}$, with $\gamma \in \Gamma$. So elements of Γ serve as coordinates of liftings with respect to the reference $\widetilde{f}$. Now for every $\gamma \in \Gamma$, the composition $\widetilde{f} \circ \gamma$ is also a lifting of f, so there is a unique $\gamma' \in \Gamma$ such that $\gamma' \circ \widetilde{f} = \widetilde{f} \circ \gamma$. This correspondence $\gamma \to \gamma'$ is determined by the reference $\widetilde{f}$, and is obviously a homomorphism.

DEFINITION 2. The endomorphism $\widetilde{f}_* : \Gamma \to \Gamma$ determined by a lifting $\widetilde{f}$ of f is defined by

$$\widetilde{f}_*(\gamma) \circ \widetilde{f} = \widetilde{f} \circ \gamma .$$

It is well known that $\Gamma \cong \pi_1(X)$. We will identify $\pi = \pi_1(X, x_0)$ and Γ in the following way. Pick base points $x_0 \in X$ and $\widetilde{x}_0 \in p^{-1}(x_0) \subset \widetilde{X}$ once for all. Now points of $\widetilde{X}$ are in 1-1 correspondence with path classes in X starting from x_0: for $\widetilde{x} \in \widetilde{X}$ take any path in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}$ and project it into X; conversely for a path c in X starting from x_0, lift it to $\widetilde{X}$ with start point at $\widetilde{x}_0$, and take its endpoint. In this way, we identify a point of $\widetilde{X}$ with a path class $\langle c\rangle$ in X starting from x_0. Under this identification $\widetilde{x}_0 = \langle e\rangle$ is the unit element in $\pi_1(X, x_0)$. The action of the loop class $\alpha = \langle a\rangle \in \pi_1(X, x_0)$ on $\widetilde{X}$ is then given by

$$\alpha = \langle a\rangle : \langle c\rangle \to \alpha \cdot \langle c\rangle = \langle a \cdot c\rangle .$$

Now, we have the following relationship between $\widetilde{f}_* : \pi \to \pi$ and

$$f_* : \pi_1(X, x_0) \to \pi_1(X, f(x_0)) .$$

Lemma 4 [9]. *Suppose $\widetilde{f}(\widetilde{x}_0) = \langle w\rangle$. Then the following diagram commutes*:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(X, f(x_0)) \\ & \searrow^{\tilde{f}_*} & \downarrow{w_*} \\ & & \pi_1(X, x_0) \end{array}$$

Lemma 5 [9]. *Lifting classes of f are in* 1-1 *correspondence with $\widetilde{f}_*$-conjugacy classes in π, the lifting class $[\gamma \circ \widetilde{f}]$ corresponding to the $\widetilde{f}_*$-conjugacy class of γ. So we have $R(f) = R(\widetilde{f}_*)$.*

We will say that the fixed point class $p(\mathrm{Fix}(\gamma \circ \widetilde{f}))$, which is labeled with the lifting class $[\gamma \circ \widetilde{f}]$, *corresponds* to the $\widetilde{f}_*$-conjugacy class of γ. Thus $\widetilde{f}_*$-conjugacy classes in π serve as coordinates for fixed point classes of f, once a reference lifting $\widetilde{f}$ is chosen.

A reasonable approach is to consider homomorphisms of π which send an $\widetilde{f}_*$-conjugacy class to one element:

Lemma 6 [9]. *The composition $\eta \circ \theta$,*

$$\pi = \pi_1(X, x_0) \xrightarrow{\theta} H_1(X) \xrightarrow{\eta} \mathrm{Coker}(H_1(X) \xrightarrow{1-f_{1*}} H_1(X)) ,$$

where θ is abelianization and η is the natural projection, sends every $\widetilde{f}_*$-conjugacy class to a single element. Moreover, any group homomorphism $\zeta : \pi \to G$ which sends every $\widetilde{f}_*$-conjugacy class to a single element, factors through $\eta \circ \theta$.

Definition 3. A map $f : X \to X$ is said to be *eventually commutative* if there exists a natural n such that $(f^n)_* \pi_1(X, x_0)$ $(\subset \pi_1(X, f^n(x_0)))$ is commutative.

By means of Lemma 4, it is easily seen that f is eventually commutative iff so is $\widetilde{f}_*$ (see [9]).

Theorem 1 yields

Theorem 6. *Suppose that the group $H_1(X, \mathbb{Z})$ is torsion free. Let f be eventually commutative and assume that no eigenvalue of $f_{1*} : H_1(X, \mathbb{Z}) \to$*

$H_1(X,\mathbb{Z})$ is a root of unity. Then the Reidemeister zeta function $R_f(z)$ is rational and

$$R_f(z) = \Big(\prod_{i=0}^{\mathrm{rg} H_1(X)} \det(E - \textstyle\bigwedge^i f_{1*}\cdot \sigma z)^{(-1)^{i+1}}\Big)^{(-1)^r} \tag{7}$$

where $\sigma = (-1)^p$, p is the number of $\mu \in \operatorname{Spec} f_{1*}$ such that $\mu < -1$ and r is the number of real $\lambda \in \operatorname{Spec} f_{1*}$ such that $|\lambda| > 1$.

EXAMPLE 1. Let $f : X \to X$ be a hyperbolic endomorphism of T^n or of a nilmanifold. Then $R_f(z)$ is a rational function and the formula (7) holds.

Theorem 2 implies

THEOREM 7. *Suppose that $f : X \to X$ is eventually commutative and $H_1(X,\mathbb{Z}) = Z_p$ (p prime). Then $R_f(z)$ is a rational function.*

COROLLARY 2. *Let $X = L(p, q_1, \ldots, q_r)$, p prime, be a generalized lens space and f as above. Then $R_f(z)$ is a rational function.*

3.1. *The Reidemeister zeta function and Serre bundles.* Let $p : E \to B$ be a Serre bundle in which E, B and every fiber are compact connected polyhedra and $F_b = p^{-1}(b)$ is a fiber over $b \in B$. A Serre bundle $p : E \to B$ is said to be (*homotopically*) *orientable* if for any two paths w, w' in B with the same endpoints $w(0) = w'(0)$ and $w(1) = w'(1)$, the fiber translations $\tau_w \cong \tau'_w : F_{w(0)} \to F_{w(1)}$. A map $f : E \to E$ is called a *fiber map* if there is an induced map $\bar f : B \to B$ such that $p \circ f = \bar f \circ p$. Let $p : E \to B$ be an orientable Serre bundle and let $f : E \to E$ be a fiber map. Then for any two fixed points b, b' of $\bar f : B \to B$, the maps $f_b = f|F_b$ and $f_{b'} = f|F_{b'}$ have the same homotopy type; hence they have the same Reidemeister numbers $R(f_b) = R(f_{b'})$ [9].

In this section we study the relation between the Reidemeister zeta functions $R_f(z)$, $R_{\bar f}(z)$ and $R_{f_b}(z)$ for a fiber map $f : E \to E$ of an orientable Serre bundle $p : E \to B$.

Theorem 4 yields

THEOREM 8. *Suppose that $f : E \to E$ admits a Fadell splitting in the sense that for some $e \in \operatorname{Fix} f$ and $b = p(e)$ the following conditions are satisfied*:

1) *the sequence*

$$0 \to \pi_1(F_b, e) \xrightarrow{i_*} \pi_1(E, e) \to \pi_1(B, b) \to 0$$

is exact,

2) *p_* admits a right inverse* (*section*) *σ such that $\operatorname{Im}\sigma$ is a normal subgroup of $\pi_1(E, e)$ and $f_*(\operatorname{Im}\sigma) \subset \operatorname{Im}\sigma$.*

Suppose $R_{\bar f}(z)$ and $R_{f_b}(z)$ are rational functions. Then so is $R_f(z)$.

3.2. *The Reidemeister zeta function of a periodic map.* Let $[\widetilde{f}]$ be a lifting class of $f : X \to X$. Then the liffting class $[\widetilde{f}^n]$ of f^n is independent of the choice of the representative $\widetilde{f}$, so we have a well-defined correspondence between the sets of conjugacy classes of liftings $\widetilde{f}$ and $\widetilde{f}^n$ such that $i([\widetilde{f}]) = [\widetilde{f}^n]$.

LEMMA 7 [9]. *Let $\widetilde{f} : \widetilde{X} \to \widetilde{X}$ be a lifting of f. Then $i([\alpha \circ \widetilde{f}]) = [\alpha^{(n)} \circ \widetilde{f}^n]$, where*

$$\alpha^{(n)} = \alpha \cdot \widetilde{f}_*(\alpha) \cdot \ldots \cdot \widetilde{f}_*^{n-1}(\alpha)\,.$$

THEOREM 9. *Suppose that $f : X \to X$ is a periodic map with least period m. Then*

$$R_f(z) = \prod_{d \mid m} \sqrt[d]{(1 - z^d)^{-\sum_{d_1 \mid d} \mu(d_1) R(f^{d/d_1})}}\,. \tag{9}$$

Proof. Let $R(f^n) = R_n$. Since $f^m = \mathrm{id}$, we have $R_j = R_{m+j}$ for every j. We show that $R_1 = R_k$ if $(k, m) = 1$. There are $t, q \in \mathbb{Z}_+$ such that $kt = mq + 1$. Then $(f^k)^t = f^{kt} = f^{mq+1} = (f^m)^q \circ f = f$. From this and Lemma 7 it follows that $\alpha_1^{(k)} \neq \alpha_2^{(k)}$ if $\alpha_1 \neq \alpha_2$ and conversely, $\alpha_1 \neq \alpha_2$ if $\alpha_1^{(k)} \neq \alpha_2^{(k)}$. Thus $R_1 = R_k$. In the same way it is proved that $R_d = R_{\mathrm{id}}$ if $(i, m/d) = 1$, where $d \mid m$. By direct calculation we hence obtain

$$\begin{aligned} R_f(z) &= \exp\left(\sum_{n=1}^{\infty} \frac{R(f^n)}{n} z^n\right) \\ &= \exp\left(\sum_{d \mid m} \sum_{n=1}^{\infty} \frac{S(d)}{d} \frac{(z^d)^n}{n}\right) = \exp\left(\sum_{d \mid m} \frac{-S(d)}{d} \ln(1 - z^d)\right) \\ &= \prod_{d \mid m} \sqrt[d]{(1 - z^d)^{-S(d)}} \end{aligned}$$

(see [4], [12] for details), where the integers $S(d)$ are calculated recursively via the formula $S(d) = R_d - \sum_{d_1 \mid d, d_1 \neq d} S(d_1)$. Moreover, if the last formula is rewritten as $R_d = \sum_{d_1 \mid d} S(d_1)$ and the Möbius Inversion Theorem is used, then $S(d) = \sum_{d_1 \mid d} \mu(d_1) R_{d/d_1}$. ■

The Mostow–Margulis rigidity theorem (see [16]) and Theorem 9 give

THEOREM 10. *Let $f : M^n \to M^n$, $n \geq 3$, be a homeomorphism of a compact hyperbolic manifold M^n. Then*

$$R_f(z) = \prod_{d \mid m} \sqrt[d]{(1 - z^d)^{-S(d)}}\,,$$

where m is the least period of the periodic map to which f is homotopic and

$$S(d)=\sum_{d_1|d}\mu(d_1)R_{d/d_1}\,.$$

§4. The computation of the Nielsen zeta function

4.1. *The Jiang subgroup and the Nielsen zeta function.* From the homotopy invariance theorem (see [9]) it follows that if a homotopy $\{h_t\}:f\cong g:X\to X$ lifts to a homotopy $\{\widetilde{h}_t\}:\widetilde{f}\cong\widetilde{g}:\widetilde{X}\to\widetilde{X}$, then we have $\operatorname{index}(f,p(\operatorname{Fix}\widetilde{f}))=\operatorname{index}(g,p(\operatorname{Fix}\widetilde{g}))$. Suppose $\{h_t\}$ is a cyclic homotopy $\{h_t\}:f\cong f$; then it lifts to a homotopy from a given lifting $\widetilde{f}$ to another lifting $\widetilde{f}'=\alpha\circ\widetilde{f}$, and we have

$$\operatorname{index}(f,p(\operatorname{Fix}\widetilde{f}))=\operatorname{index}(f,p(\operatorname{Fix}\alpha\circ\widetilde{f}))\,.$$

In other words, a cyclic homotopy induces a permutation of lifting classes (hence of fixed point classes); those in the same orbit of this permutation have the same index. This idea is applied to the computation of $N_f(z)$.

Definition 4. The *trace subgroup of cyclic homotopies* (the *Jiang subgroup*) $I(\widetilde{f})\subset\pi$ is defined by $I(\widetilde{f})=\{\alpha\in\pi|$ there exists a cyclic homotopy $\{h_t\}:f\simeq f$ which lifts to $\{\widetilde{h}_t\}:\widetilde{f}\cong\alpha\circ\widetilde{f}\}$ (see [9]).

Let $Z(G)$ denote the center of a group G, and let $Z(H,G)$ denote the centralizer of a subgroup $H\subset G$. The Jiang subgroup has the following properties:

1) $I(\widetilde{f})\subset Z(\widetilde{f}_*(\pi),\pi)$;
2) $I(\mathrm{id}_{\widetilde{X}})\subset Z(\pi)$;
3) $I(\widetilde{g})\subset I(\widetilde{g}\circ\widetilde{f})$;
4) $\widetilde{g}_*(I(\widetilde{f}))\subset I(\widetilde{g}\circ\widetilde{f})$;
5) $I(\mathrm{id}_{\widetilde{X}})\subset I(\widetilde{f})$.

The class of path-connected spaces X satisfying the condition $I(\mathrm{id}_{\widetilde{X}})=\pi=\pi_1(X,x_0)$ is closed under homotopy equivalence and the topological product operation, and contains the simply connected spaces, generalized lens spaces, H-spaces, homogeneous spaces of the form G/G_0 where G is a topological group and G_0 a subgroup which is a connected compact Lie group (for the proofs see [9]).

Theorem 11. *Suppose that* $\widetilde{f}_*(\pi)\subset I(\widetilde{f})$ *and* $L(f^n)\neq0$ *for every* $n>0$. *Then*

$$N_f(z)=R_f(z)=\exp\left(\sum_{n=1}^{\infty}\frac{\operatorname{ord}\operatorname{Coker}(1-f^n_{1*})}{n}z^n\right). \tag{10}$$

P r o o f. We have $\widetilde{f}^n_*(\pi)\subset I(\widetilde{f}^n)$ for every $n>0$ (by property 4) and the condition $\widetilde{f}_*(\pi)\subset I(\widetilde{f})$). For any $\alpha\in\pi$, $p(\operatorname{Fix}\alpha\circ\widetilde{f}^n)=p(\operatorname{Fix}\widetilde{f}^n_*(\alpha)\circ\widetilde{f}^n)$ by

Lemmas 3 and 5. Since $\widetilde{f}_*^n(\pi) \subset I(\widetilde{f}^n)$, there is a homotopy $\{h_t\} : f^n \cong f^n$ which lifts to $\{\widetilde{h}_t\} : \widetilde{f}^n \cong \widetilde{f}_*^n(\alpha) \circ \widetilde{f}^n$. Hence $\operatorname{index}(f^n, p(\operatorname{Fix} \widetilde{f}^n)) = \operatorname{index}(f^n, p(\operatorname{Fix} \alpha \circ \widetilde{f}^n))$. Since $\alpha \in \pi$ is arbitrary, any two fixed point classes of f^n have the same index. It immediately follows that $L(f^n) = 0$ implies $N(f^n) = 0$ and $L(f^n) \neq 0$ implies $N(f^n) = R(f^n)$. By property 1), $\widetilde{f}^n(\pi) \subset I(\widetilde{f}^n) \subset Z(\widetilde{f}_*^n(\pi), \pi)$, so $\widetilde{f}_*^n(\pi)$ is abelian. Hence $\widetilde{f}_*^n$ is eventually commutative and $R(f^n) = \operatorname{ord} \operatorname{Coker}(1 - f_{1*}^n)$. ■

R e m a r k 1. The conclusion of Theorem 11 remains valid if we use the condition "there is an integer m such that $\widetilde{f}_*^m(\pi) \subset I(\widetilde{f}^m)$" instead of the stronger condition $\widetilde{f}_*(\pi) \subset I(\widetilde{f})$, but the proof is more complicated.

COROLLARY 4. *Let* $I(\mathrm{id}_{\widetilde{X}}) = \pi$ *and* $L(f^n) \neq 0$ *for every* $n > 0$. *Then the formula* (10) *is valid.*

COROLLARY 5. *Suppose that* X *is aspherical*, f *is eventually commutative and* $L(f^n) \neq 0$ *for every* $n > 0$. *Then the formula* (10) *is valid.*

THEOREM 12. *Suppose that* $H_1(X, \mathbb{Z})$ *is torsion-free and there exists an integer* m *such that* $\widetilde{f}_*^m(\pi) \subset I(\widetilde{f}^m)$. *Let* $L(f^n) \neq 0$ *for every* $n > 0$. *Then the Nielsen zeta function* $N_f(z)$ *is rational and*

$$N_f(z) = R_f(z) = \Big(\prod_{i=0}^{\operatorname{rg} H_1(X)} \det(E - \textstyle\bigwedge^i f_{1*} \cdot \sigma z)^{(-1)^{i+1}} \Big)^{(-1)^r} \tag{11}$$

where σ *and* r *are the same as in Theorem* 6.

P r o o f. From the assumptions of the theorem it follows that for every $n > 0$

$$\begin{aligned} 0 \neq N(f^n) = R(f^n) &= \operatorname{ord} \operatorname{Coker}(1 - f_{1*}^n) = |\det(E - f_{1*}^n)| \\ &= (-1)^{r+pn} \det(E - f_{1*}^n)\,. \end{aligned}$$

Thus we have the "trace formula" for the Nielsen numbers:

$$N(f^n) = (-1)^{r+pn} \sum_{i=0}^{\operatorname{rg} H_1(X)} (-1)^i \operatorname{tr} \textstyle\bigwedge^i f_{1*}^n\,. \tag{12}$$

Now (11) follows from a calculation as in Theorem 1. ■

COROLLARY 6. *Suppose that the assumptions of Theorem* 12 *hold. Then the functional equation for the Nielsen zeta function* $N_f(z)$ *is*

$$N_f\left(\frac{1}{dz}\right) = (N_f(z))^{(-1)^{\operatorname{rg} H_1(X)}} \cdot \varepsilon\,, \tag{13}$$

where $d = \det(f_{1*})$, $\varepsilon \in \mathbb{C}$.

EXAMPLE 2. Let $f : T^n \to T^n$ be a hyperbolic endomorphism of T^n. Then $N_f(z) = R_f(z)$ is rational and the formulas (11–13) hold. In this case $d = \det(f_{1*})$ is the degree of f.

COROLLARY 7. *Under the hypotheses of Theorem* 12 *the poles and zeros of the Nielsen zeta function are complex numbers reciprocal to the eigenvalues of the matrices* $\bigwedge^i f_{1*} \cdot \sigma$, $0 \le i \le \operatorname{rg} H_1(X, \mathbb{Z})$.

4.2. *Polyhedra with finite fundamental group.* For a compact polyhedron X with finite fundamental group $\pi_1(X)$, the universal covering space $\widetilde{X}$ is compact, so that we can explore the relation between $L(\widetilde{f})$ and $\operatorname{index}(p(\operatorname{Fix} \widetilde{f}))$.

DEFINITION 5 [9]. The number $\mu([\widetilde{f}^n]) = \# \operatorname{Fix} \widetilde{f}^n_*$, the order of the fixed-element group $\operatorname{Fix} \widetilde{f}^n_*$, is called the *multiplicity* of the lifting class $[\widetilde{f}^n]$, or of the fixed point class $p(\operatorname{Fix} \widetilde{f}^n)$.

LEMMA 8 [9]. $L(\widetilde{f}^n) = \mu([\widetilde{f}^n]) \cdot \operatorname{index}(f^n, p(\operatorname{Fix} \widetilde{f}^n))$.

LEMMA 9 [9]. *If* $R(f^n) = \operatorname{ord} \operatorname{Coker}(1 - f^n_{1*})$ (*in particular, if* f *is eventually commutative*), *then*

$$\mu([\widetilde{f}^n]) = \operatorname{ord} \operatorname{Coker}(1 - f^n_{1*}) \, .$$

THEOREM 13. *Let* X *be a connected compact polyhedron with finite fundamental group* π. *Suppose that the action of* π *on the rational homology of the universal covering space* $\widetilde{X}$ *is trivial*, *i.e. for every covering translation* $\alpha \in \pi$, $\alpha_* = \operatorname{id} : H_*(\widetilde{X}, \mathbb{Q}) \to H_*(\widetilde{X}, \mathbb{Q})$. *Let* f *be eventually commutative and* $L(f^n) \ne 0$ *for every* $n > 0$. *Then*

$$N_f(z) = R_f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{\operatorname{ord} \operatorname{Coker}(1 - f^n_{1*})}{n} z^n \right) . \tag{14}$$

Proof. Under our assumption on X any two liftings $\widetilde{f}$ and $\alpha \circ \widetilde{f}$ induce the same homology homomorphism $H_*(\widetilde{X}, \mathbb{Q}) \to H_*(\widetilde{X}, \mathbb{Q})$, hence the same $L(\widetilde{f})$. Then from Lemma 8 it follows that any two fixed point classes are either both essential or both inessential. The statement is now a consequence of Lemma 9. ∎

LEMMA 10 [9]. *Let* X *be a polyhedron with finite fundamental group* π *and let* $p : \widetilde{X} \to X$ *be its universal covering. Then the action of* π *on the rational homology of* $\widetilde{X}$ *is trivial iff* $H_*(\widetilde{X}, \mathbb{Q}) \cong H_*(X, \mathbb{Q})$.

COROLLARY 8. *Let* $\widetilde{X}$ *be a compact* 1*-connected polyhedron which is a rational homology* n*-sphere*, n *odd. Let* π *be a finite group acting freely on* $\widetilde{X}$, *and* $X = \widetilde{X}/\pi$. *Then Theorem* 13 *applies.*

Proof. The projection $p : \widetilde{X} \to X = \widetilde{X}/\pi$ is a universal covering space of X. For every $\alpha \in \pi$, the degree of $\alpha : \widetilde{X} \to \widetilde{X}$ must be 1, because $L(\alpha) = 0$ (α has no fixed points). Hence $\alpha_* = \mathrm{id} : H_*(\widetilde{X}, \mathbb{Q}) \to H_*(\widetilde{X}, \mathbb{Q})$. ∎

COROLLARY 9. *If X is a closed 3-manifold with finite π, then Theorem 13 applies.*

Proof. $\widetilde{X}$ is an orientable simply connected manifold, hence a homology 3-sphere. Apply Corollary 8. ∎

§ 5. Concluding remarks, problems, examples

5.1. *"Entropy conjecture" for the Reidemeister numbers and the radius of convergence R for the Reidemeister zeta function.* Let $h(f)$ be the topological entropy of f and set $h = \inf h(g)$, infimum being taken over all maps g of the homotopy type of f.

THEOREM 14. *Let the assumptions of Theorem 11 or 13 hold. Then*

$$h(f) \geq \limsup_{n\to\infty} \frac{1}{n} \log R(f^n) \geq 0 \quad \text{and} \quad 1 \geq R \geq e^{-h} > 0\,.$$

Proof. The statement follows from N.V. Ivanov's inequality [8]

$$h(f) \geq \limsup_{n\to\infty} \frac{1}{n} \log N(f^n)\,,$$

the Cauchy–Hadamard formula and the homotopy invariance of R. ∎

PROBLEM. For what maps f the inequality

$$h(f) \geq \limsup_{n\to\infty} \frac{1}{n} \log R(f^n)$$

holds?

5.2. *Examples.* Let $f : X \to X$ be a continuous map of a simply connected compact polyhedron. Then $R_f(z) = 1/(1-z)$.

For the next example, let $\rho : M \to M$ be an expanding map of an orientable compact smooth manifold [14]. Then $R_\rho(z)$ and $N_\rho(z)$ are rational functions and $R_\rho(z) = N_\rho(z) = L_\rho(\sigma z)^{(-1)^r}$, where $r = \dim M$, $\sigma = +1$ if ρ preserves the orientation of M, and $\sigma = -1$ if ρ reverses the orientation of M (see [12]).

In particular, if $f : S^1 \to S^1$ is a continuous map of degree d, $|d| \neq 1$, then $R_f(z) = N_f(z) = (1-z)/(1-dz)$ if $d > 0$; $R_f(z) = N_f(z) = 1/(1-z)$ if $d - 0$; and $R_f(z) = N_f(z) = (1+z)/(1+dz)$ if $d < 0$.

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DEPARTMENT OF MATHEMATICS
LENINGRAD TECHNOLOGY INSTITUTE
MOSKOVSKIĬ PROSP. 26
LENINGRAD 198013, U.S.S.R.

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