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POLYHEDRAL QUOTIENT SPACES

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Throughout this paper, P will be an arbitrary polyhedron (all our polyhedra are compact), $E \subset P \times P$ will be an equivalence relation on P, and q will denote the quotient projection of P onto the quotient space P/E. We are interested in the following question: when is P/E a polyhedron such that q is PL? More precisely, when does there exist a polyhedron Q and a PL map $f : P \to Q$ inducing a homeomorphism $P/E \to Q$? We shall answer this question for the case that the equivalence classes of E (the fibers of q) are finite sets.

Notice that a pair (Q, f) satisfying the condition of the preceding paragraph – if such a pair exists – possesses the following universal property: for every polyhedron Q' and every PL map $f' : P \to Q'$ such that $(x, y) \in E$ implies $f'(x) = f'(y)$, the unique map $g : Q \to Q'$ satisfying $f' = gf$ is PL. It follows that such a pair (Q, f) is essentially unique and that it can be considered the quotient of P (with respect to E) in the PL category. Therefore, when P/E admits a PL structure such that q is PL we shall say that P/E is a *PL quotient space* of P.

It can easily happen that P/E admits polyhedral structures, but none with q being PL; e.g. the quotient space obtained by shrinking a subpolyhedron of P to a point is always a topological polyhedron but rarely a PL quotient space because a linear map cannot shrink a face of a simplex to a point and be injective on the rest of the simplex. By similar consideration we can convince ourselves that there are hardly any interesting PL quotient spaces with degenerate quotient projection. Therefore we shall from now on restrict our attention to the case that q is nondegenerate, i.e. the equivalence classes of E are finite sets. For future reference we state the following well-known triangulation criterion for PL quotient spaces.

PROPOSITION 1. *Suppose that E has finite equivalence classes. P/E is a PL quotient space if and only if there exist a triangulation K of P and a labeling of the vertices of K such that*

(a) *the endpoints of any* 1*-simplex of K are assigned different labels and*

(b) *for arbitrary points* $x, y \in P$ *the following equivalence holds*: $(x, y) \in E$ *if and only if there exist simplices* $\sigma, \tau \in K$ *with* $x \in \sigma$ *and* $y \in \tau$ *and there exists a label preserving simplicial isomorphism* $\sigma \to \tau$ *taking* x *to* y.

This proposition is difficult to apply directly except in the most simple concrete cases. We shall give some criteria that are easier to apply.

Under a *partial PL homeomorphism* of P we shall understand a PL homeomorphism between two subpolyhedra of P. Analogously we define a partial simplicial isomorphism of a simplicial complex. The domain of a partial PL homeomorphism f will be denoted by $D(f)$. A set F of partial PL homeomorphisms of P will be said to *generate* E if E is the smallest equivalence relation on P containing all pairs $(x, f(x))$ for f in F and x in $D(f)$.

Arbitrary partial homeomorphisms f and g of P are composed as relations on P: if the preimage $f^{-1}(D(g))$ is nonempty, then by gf (or $g \circ f$) we mean the composite $g \circ (f|f^{-1}D(g))$ in the usual sense; if $f^{-1}(D(g)) = \emptyset$, we set $gf := \emptyset$.

Theorem 2. *The following assertions are equivalent*:

(a) *E has finite equivalence classes*, *and P/E is a PL quotient space.*

(b) *There exist a subpolyhedron $X \subset P$, a polyhedron Y, and a nondegenerate PL map $g : X \to Y$ such that $(x_1, x_2) \in E$ if and only if either $x_1 = x_2$ or $x_1, x_2 \in X$ and $g(x_1) = g(x_2)$.*

(c) *E is generated by a finite set of partial simplicial isomorphisms of a triangulation of P.*

(d) *E is generated by a finite set F of partial PL homeomorphisms of P such that the collection of all possible composites of members of F and their inverses is finite.*

(e) *E is generated by a finite set F of partial PL homeomorphisms of P, and there exists another finite set G of partial PL homeomorphisms of P such that every composite of members of F and/or their inverses is a restriction of a member of G.*

(f) *E is a subpolyhedron of $P \times P$, and the first* (*and hence the second*) *natural projection from $P \times P$ to P is nondegenerate on E.*

We give two sample applications; no originality is claimed (concerning Corollary 3 see Rourke and Sanderson [2; Problem 2.27(4)] and Bredon [1; Chapter III, §1]).

Corollary 3. *If a finite group G acts on P through PL homeomorphisms*, *then the orbit space P/G is a PL quotient space.*

Corollary 4. *If X*, *Y and $A \subset X$ are polyhedra and $g : A \to Y$ is a nondegenerate PL map*, *then $X \cup_g Y$ is a PL quotient space of the* *topological disjoint union* $X+Y$. *In particular*, *the mapping cylinder of a nondegenerate PL map is a PL quotient space.*

For a while I believed that the following common weakening of the statements (c)–(e) in Theorem 2 is equivalent to (a)–(f): E has finite equivalence classes and is generated by a finite set of partial PL homeomorphisms of P. But then I found the following simple counterexample. Let P be the unit square $[0,1]^2$, let $A:\mathbb{R}^2\to\mathbb{R}^2$ be the linear map defined by $A(x,y):=(x+y,y)$, and let f be the restriction of A to $P\cap A^{-1}(P)$. Then the equivalence relation E on P generated by f has finite equivalence classes (= intersections of P with the orbits of A on $\mathbb{R}^2$), but P/E is not even a Hausdorff space.

Proof of Theorem 2. (a)⇒(b). Let $X:=P$, $Y:=P/E$, and $g:=q$.

(b)⇒(c). Choose a triangulation K of P such that X is triangulated by a subcomplex L of K and g is simplicial from L to some triangulation of Y. For each pair of simplices $\sigma,\tau\in L$ with the same image under g we have the simplicial isomorphism $\varphi_{\sigma\tau}:=(g|\tau)^{-1}(g|\sigma):\sigma\to\tau$, and the collection of all such isomorphisms generates E.

The implication (c)⇒(d) is obvious.

(d)⇒(f). We may assume that the (finite) family F contains id_P, the inverses of all its members, and all possible composites of its members. Then $(x,y)\in E$ if and only if there exists an $f\in F$ such that $x\in D(f)$ and $y=f(x)$, i.e. E is the union of the graphs of all $f\in F$. Since the graph of a PL map is a polyhedron, E is a polyhedron, and clearly the first projection $P\times P\to P$ is nondegenerate on E.

(f)⇒(e). Denote by $\pi_1,\pi_2:E\to P$ the restrictions of the two natural projections $P\times P\to P$. Choose triangulations K for E and L for P such that π_1 is simplicial from K to L. For each simplex $\sigma\in K$, the map

$$g_\sigma:=\pi_2\circ(\pi_1|\sigma)^{-1}:\pi_1(\sigma)\to\pi_2(\sigma)$$

is a linear embedding of the simplex $\pi_1(\sigma)\in L$ into P. Denote by G the set of all g_σ $(\sigma\in K)$.

Let $j:E\to E$ be the involution defined by $j(x,y):=(y,x)$. For each pair of simplices $\sigma,\tau\in K$ such that $\sigma\cap j(\tau)\neq\emptyset$ let

$$f_{\sigma\tau}:=\pi_2\circ(\pi_1|\sigma\cap j(\tau))^{-1}=g_\sigma|\pi_1(\sigma\cap j(\tau))$$

and let F be the family of all such maps $f_{\sigma\tau}$. Then F generates E: for an arbitrary point $(x,y)\in E$ there are simplices $\sigma,\tau\in K$ such that $(x,y)\in\sigma$ and $(y,x)\in\tau$, and then $x\in D(f_{\sigma\tau})$ and $y=f_{\sigma\tau}(x)$.

Each $f_{\sigma\tau}\in F$ is a restriction of $g_\sigma\in G$, and its inverse $f_{\sigma\tau}^{-1}=f_{\tau\sigma}$ is a restriction of $g_\tau\in G$. Therefore, to prove that E satisfies condition (e) of Theorem 2 it suffices to show that the composite of any two members of G is a restriction of a member of G.

Take arbitrary simplices $\sigma, \tau \in K$ such that $D(g_\tau g_\sigma) \neq \emptyset$. The map $s : D(g_\tau g_\sigma) \to P$ defined by $s(x) := (x, g_\tau g_\sigma(x))$ maps into E and is thus a section of π_1. Since $D(g_\tau g_\sigma) = \pi_1(\sigma) \cap g_\sigma^{-1}(\pi_1(\tau))$ is a convex subset of a simplex of L and since π_1 is a nondegenerate simplicial map from K to L, the image of s lies in a simplex ρ of K. Therefore s is a restriction of $(\pi_1|\rho)^{-1}$ and $g_\tau g_\sigma = \pi_2 s$ is a restriction of g_ρ.

(e)⇒(d). We may assume that $\mathrm{id}_P \in G$ and that $f^{-1} \in F$ for each $f \in F$. Denote by H the family of all possible composites of members of F. We will show that H is finite. Take any $h \in H$ and any $f \in F$. By hypothesis there exists a $g \in G$ such that h is a restriction of g. Then $D(fh) = D(h) \cap h^{-1}(D(f)) = D(h) \cap g^{-1}(D(f))$. By induction on the length of composites it follows that the family $\{D(h) \mid h \in H\}$ is contained in the family Δ of all possible intersections

$$\bigcap_{k=1}^{n} g_k^{-1}(D(f_k))$$

where n is any positive integer and $f_k \in F$ and $g_k \in G$ for $k = 1, \ldots, n$. But Δ is finite, and therefore H is contained in the finite family of all restrictions $g|D$ for $g \in G$ and $D \in \Delta$.

(d)⇒(c). We may assume that the (finite) family F contains id_P, the inverses of all its members, and all possible composites of its members. For each $x \in P$ let F_x be the set of all $f \in F$ such that $x \in D(f)$. Choose a triangulation T of P such that for each $f \in F$ the following holds: $D(f)$ is triangulated by a subcomplex of T, and f is linear on each simplex of $T|D(f)$. For every $x \in P$ let $C(x)$ denote the carrier of x in T, i.e. the smallest simplex of T containing x.

For every $x \in P$ let

$$\sigma_x := \bigcap_{f \in F_x} f^{-1}(C(f(x))) \tag{1}$$

and let $S := \{\sigma_x \mid x \in P\}$. Clearly S is a finite family of nonempty subsets of P. We assert that

(i) each $\sigma_x \in S$ is a convex linear cell lying in $C(x)$, and

$$x \in \operatorname{int} \sigma_x = \bigcap_{f \in F_x} f^{-1}(\operatorname{int} C(f(x))) \subset \operatorname{int} C(x); \tag{2}$$

(ii) if $\sigma, \tau \in S$ and $\sigma \neq \tau$, then $(\operatorname{int} \sigma) \cap (\operatorname{int} \tau) = \emptyset$;

(iii) for each $\sigma, \tau \in S$, $\sigma \cap \tau$ is a union of cells in S;

(iv) each $f \in F$ maps each cell of S (on which it is defined) linearly onto some cell of S.

P r o o f of (i). As $\mathrm{id} \in F_x$, the intersection (1) contains the term $C(x)$; thus $\sigma_x \subset C(x)$. Suppose that $\mathbb{R}^n$ is the ambient Euclidean space and let

$\Pi \subset \mathbb{R}^n$ be the plane spanned by $C(x)$. For each $f \in F_x$ there is an injective linear map $f' : \Pi \to \mathbb{R}^n$ such that f and f' agree on $C(x)$. Hence

$$\sigma_x = \bigcap_{f \in F_x} f^{-1}(C(f(x))) = \bigcap_{f \in F_x} f'^{-1}(C(f(x)))$$

(again we use the fact that $C(x)$ appears as a term in both intersections). It clearly follows that σ_x is a convex linear cell. Similarly we have

$$\bigcap_{f \in F_x} f^{-1}(\operatorname{int} C(f(x))) = \bigcap_{f \in F_x} f'^{-1}(\operatorname{int} C(f(x))).$$

Since the last written intersection is nonempty (it contains x) it is equal to $\bigcap_f \operatorname{int} f'^{-1}(C(f(x)))$ and further to $\operatorname{int}(\bigcap_f f'^{-1}(C(f(x)))) = \operatorname{int} \sigma_x$, which proves (2).

Proof of (ii). Suppose that $x \in (\operatorname{int} \sigma_a) \cap (\operatorname{int} \sigma_b)$. Then, by (2), x lies in $(\operatorname{int} C(a)) \cap (\operatorname{int} C(b))$, and hence $C(a) = C(b)$. Every $f \in F$ which is defined at a or b is defined on the whole simplex $C(a) = C(b)$, and therefore $F_a = F_b$. For each $f \in F_a = F_b$ we have, by (i), $f(x) \in \operatorname{int} C(f(a))$ and $f(x) \in \operatorname{int} C(f(b))$, which implies $C(f(a)) = C(f(b))$, and now it follows from (1) that $\sigma_a = \sigma_b$.

Proof of (iii). Suppose that $x \in \sigma_a \cap \sigma_b$. It obviously follows from (1) that $F_x \supset F_a \cup F_b$. The relation $x \in \sigma_a$ implies that for each $f \in F_a$ we have $f(x) \in C(f(a))$ and therefore $C(f(x)) \subset C(f(a))$. Similarly we have $C(f(x)) \subset C(f(b))$ for each $f \in F_b$. Hence

$$x \in \sigma_x \subset \bigcap_{f \in F_a \cup F_b} f^{-1}(C(f(x))) \subset \sigma_a \cap \sigma_b.$$

Proof of (iv). Suppose that $f \in F$ is defined on $\sigma = \sigma_x \in S$. It follows from (i) that $f|\sigma$ is a linear embedding, and it remains to prove that $\tau := f(\sigma) \in S$. We assert that $\tau = \sigma_y$ where $y := f(x)$.

Let $F_x =: \{g_1, \ldots, g_r\}$. Then

$$\tau = f\Big(\bigcap_{i=1}^{r} g_i^{-1}(C(g_i(x)))\Big) = \bigcap_{i=1}^{r} f(g_i^{-1}C(g_i(x))).$$

For each i $(= 1, \ldots, r)$, the composite $h_i := g_i f^{-1}$ belongs to F and in fact to F_y. Therefore

$$\tau = \bigcap_{i=1}^{r} h_i^{-1}(C(h_i(y))) \supset \sigma_y.$$

In order to prove that $\tau \subset \sigma_y$ we take an arbitrary $h \in F_y$ and show that $\tau \subset h^{-1}(C(h(y)))$. Since $g := hf \in F_x$ there is an $i \in \{1, \ldots, r\}$ such that $g = g_i$. It follows that $h_i = g_i f^{-1} = hff^{-1}$ is a restriction of h, and therefore $\tau \subset h_i^{-1}(C(h_i(y))) \subset h^{-1}(C(h(y)))$ as asserted.

By (i), (ii), and (iii), S is a cell complex with all cells convex and linear. Therefore S has a well defined barycentric subdivision, K, which is a simplicial complex. It obviously follows from (iv) that each $f \in F$ maps each simplex of $K|D(f)$ linearly onto a simplex of K, i.e. f is a partial simplicial isomorphism of K.

(c)$\Rightarrow$(a). Let K be a triangulation of P and F be a finite family of partial simplicial isomorphisms of K which generates E. We may assume that F contains the inverses of all its members and all the composites of its members. Then E-equivalence classes in P coincide with "orbits" with respect to F, i.e. for arbitrary $x, y \in P$ the following is true: $(x, y) \in E$ if and only if there is an $f \in F$ such that $x \in D(f)$ and $f(x) = y$ (clearly in this case $D(f)$ contains the whole carrier of x in K).

Denote by K' and K'' the first and second barycentric subdivisions of K. Clearly each $f \in F$ is a partial simplicial selfisomorphism of K' and of K''. We shall prove (a) by showing that K'' (in the role of K) satisfies the hypotheses of Proposition 1, where, of course, we take one label for each E-equivalence class (= F-orbit) of vertices of K''. First we observe the following:

$(*)$ *No two distinct points lying in the same simplex of* K' *are* E*-equivalent.*

Indeed, take any simplex $\sigma' \in K'$ and let σ be the smallest simplex of K (of dimension n, say) containing σ'. There is an ordering of the vertices of σ such that with respect to this ordering the barycentric coordinates (in σ) of any point of σ' form a nondecreasing sequence. It follows that the *unordered* $(n+1)$-tuples, not just sequences, of the barycentric coordinates (in σ) of any two distinct points $x, y \in \sigma'$ are distinct, and therefore no $f \in F$ can map x to y.

By $(*)$, the endpoints of no 1-simplex of K'' carry the same label. i.e. K'' satisfies condition (a) of Proposition 1. To check condition (b) take an arbitrary point $(x, y) \in E$ and let $\sigma, \tau \in K$ be the carriers of x and y, respectively. There is an $f \in F$ defined on σ such that $f(\sigma) = \tau$ and $f(x) = y$. If σ'' is the carrier of x in K'' and $\tau'' := f(\sigma'')$, then $y \in \tau'' \in K''$ and $f\,|\,\sigma'' : \sigma'' \to \tau''$ is a label preserving simplicial isomorphism sending x to y.

To prove the converse (i.e. that each label preserving simplicial isomorphism maps each point to an E-equivalent point) it obviously suffices to show that any label preserving simplicial isomorphism $\varphi : \sigma'' \to \tau''$ (for any $\sigma'', \tau'' \in K''$) is the restriction of some $f \in F$. Let σ' and σ be the carriers of σ'' in K' and K, respectively. Denote by v the barycenter of σ' (a vertex of σ'') and let $w := \varphi(v)$ (a vertex of τ''). Since $(v, w) \in E$ (as φ preserves labels) and $v \in \operatorname{int} \sigma$ there is an $f \in F$ such that $\sigma \subset D(f)$ and $f(v) = w$. Let $\tau' := f(\sigma') \in K'$. Since the vertex w of τ'' is the barycenter of τ' the whole τ'' lies in τ'. Now, for each vertex u of σ'', the points $\varphi(u)$ and $f(u)$ lie in τ' and are E-equivalent; therefore $(*)$ implies that $\varphi(u) = f(u)$, which proves that $\varphi = f \mid \sigma''$.

REFERENCES

[1] G. E. Bredon, *Introduction to Compact Transformation Groups*, Pure and Appl. Math. 46, Academic Press, New York 1972.

[2] C. P. Rourke and B. J. Sanderson, *Introduction to Piecewise Linear Topology*, Ergeb. Math. Grenzgeb. 69, Springer, Berlin 1972.

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