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A NOTE ON THE ALMOST EVERYWHERE CONVERGENCE OF ALTERNATING SEQUENCES WITH DUNFORD–SCHWARTZ OPERATORS

BY

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1. Introduction. Let L_p, $1 \le p \le \infty$, be the usual Banach spaces of real or complex functions on a σ-finite measure space $(X, \mathfrak{F}, \mu)$. By a *Dunford–Schwartz operator* we mean a linear operator T which maps the linear space $L_1 + L_\infty$ into itself and is a contraction of L_p into L_p for each $1 \le p \le \infty$ (i.e. $\|Tf\|_p \le \|f\|_p$ for all $f \in L_p$), and satisfies

$$Tf = \lim_n Tf_n \quad \text{almost everywhere}$$

whenever (f_n) is a sequence in L_∞, $f = \lim_n f_n$ almost everywhere and $\sup_n \|f_n\|_\infty < \infty$. The following is known (see e.g. [9], [10]): If T is a linear contraction of L_1 into L_1 and satisfies $\|Tf\|_\infty \le \|f\|_\infty$ for all $f \in L_1 \cap L_\infty$, or if T is a linear operator mapping $\bigcup_{1<p<\infty} L_p$ into itself and is a contraction of L_p into L_p for each $1 < p < \infty$, then T can be uniquely extended to a Dunford–Schwartz operator.

In this note we deal with a sequence (T_n) of Dunford–Schwartz operators on $L_1 + L_\infty$ and discuss the almost everywhere convergence of the alternating sequence

$$T_1^* \ldots T_n^* T_n \ldots T_1 f \quad (f \in L_1 + L_\infty)\,.$$

Using an approximation argument involving maximal operators and a result of Akcoglu [1] which states that if $f \in L_p$, $l < p < \infty$, then the alternating sequence converges almost everywhere, we shall prove that if $f \in L_1 + L_\infty$ satisfies

$$\int |f| \log^+(|f|/a)\, d\mu < \infty \quad \text{for all } a > 0\,,$$

then the alternating sequence converges almost everywhere; thus a generalization of Akcoglu's result will be obtained.

It should be noted here that a similar result has been announced in Assani [3]; but we could not see the details. (After the first manuscript of this paper was submitted, the author could get Assani's paper *Rota's alternating procedure with non-positive operators* (to appear in Adv. in Math.), in which Assani deals with Dunford–Schwartz operators defined on the *real* linear space L_1 of a *finite measure space*. The author thinks that Assani's paper does not include the result of this note.)

2. Result

THEOREM. *Let* (T_n) *be a sequence of Dunford–Schwartz operators on* $L_1 + L_\infty$ *and let* $f \in L_1 + L_\infty$ *be such that*

$$\int |f| \log^+(|f|/a)\, d\mu < \infty \qquad \textit{for all } a > 0\,.$$

Then $\lim_n T_1^* \ldots T_n^* T_n \ldots T_1 f$ *exists a.e. on* X.

The theorem does not hold if f is only assumed to be in L_1; an example was given by Burkholder [4]. In case $\mu(X) = \infty$, it may happen that there exists a function f in $L_1 + L_\infty$ which satisfies the condition of the theorem but is not in L_1; an example can be found in Fava [6]. As is easily seen, each f in L_p, $1 < p < \infty$, satisfies the condition of the theorem.

Proof. It suffices to consider the case $f \geq 0$. Given an $\varepsilon > 0$, put

$$e = f \cdot 1_{\{f \leq \varepsilon\}} \quad \text{and} \quad g = f - e$$

where 1_A denotes the indicator of a set A, and write

$$\begin{cases} f_n = T_1^* \ldots T_n^* T_n \ldots T_1 f \\ e_n = T_1^* \ldots T_n^* T_n \ldots T_1 e \quad (n \geq 1). \\ g_n = T_1^* \ldots T_n^* T_n \ldots T_1 g \end{cases} \tag{1}$$

It follows that

$$f_n = e_n + g_n \quad \text{and} \quad \|e_n\|_\infty \leq \|e\|_\infty \leq \varepsilon \quad (n \geq 1)\,. \tag{2}$$

Since $\mu(\{g > 0\}) = \mu(\{f > \varepsilon\}) < \infty$, we then have $g \in L_1$ and further $\int g \log^+ g d\mu < \infty$.

We now choose $0 < h \in L_1$ with $1 \geq h \geq \min\{g, 1\}$, and apply Doob's [5] and Starr's [10] argument as follows. First, let τ_n denote the linear modulus of T_n (see e.g. [7], p. 159); thus τ_n is a *positive* Dunford–Schwartz operator on $L_1 + L_\infty$ satisfying $|T_n f| \leq \tau_n |f|$ for all $f \in L_1 + L_\infty$. By Lemma 2 in [10], setting $\widetilde{g} = g/h$ there exist finite measure spaces (X_k, μ_k), $k = 0, 1, \ldots$, for which $X \subset X_k$, $X = X_0$, $\mu_0 = h\, d\mu$, and positive linear operators S_k from $L_1(X_{k-1}, \mu_{k-1})$ to $L_1(X_k, \mu_k)$ for which $S_k 1 = 1$ a.e. (μ_k), $S_k^* 1 = 1$ a.e. (μ_{k-1}) and

$$S_1^* \ldots S_k^* [(\tau_k \ldots \tau_1 h)(S_k \ldots S_1 \widetilde{g})] = \tau_1^* \ldots \tau_k^* \tau_k \ldots \tau_1 g \quad \text{a.e. } (\mu_0)\,. \tag{3}$$

Since $\widetilde{g} h = g$ and $\log^+ \widetilde{g} = \log^+ g$, it follows that

$$\int \widetilde{g} \log^+ \widetilde{g}\, d\mu_0 = \int \widetilde{g} (\log^+ \widetilde{g}) h\, d\mu = \int g \log^+ g\, d\mu < \infty\,. \tag{4}$$

We next choose a sequence (r_t), $t = 1, 2, \ldots$, of functions in L_2 such that $0 \le r_t \uparrow g$ a.e. on X, and write

$$\widetilde{r}_t = (g - r_t)/h\,.$$

From (3) and the fact that $0 < h \le 1$ it follows that

$$(5)\qquad S_1^* \ldots S_k^* S_k \ldots S_1 \widetilde{r}_t \ge \tau_1^* \ldots \tau_k^* \tau_k \ldots \tau_1 (g - r_t) \quad \text{a.e. } (\mu_0)\,.$$

Further, from [5] or [10], if the usual probability notation is used, we may write

$$(6)\qquad S_1^* \ldots S_k^* S_k \ldots S_1 \widetilde{r}_t = E\{E\{\widetilde{r}_t(x_0) \,|\, x_k\} \,|\, x_0\} \quad \text{a.e. } (P)\,,$$

and

$$(7)\qquad S_k \ldots S_1 \widetilde{r}_t = E\{\widetilde{r}_t(x_0) \,|\, x_k\} = E\{\widetilde{r}_t(x_0) \,|\, x_k, x_{k+1}, \ldots\} \quad \text{a.e. } (P)$$

where x_k is the kth coordinate function on the product space $\Omega = X_0 \times X_1 \times \ldots$ and P is the finite measure on Ω defined to make the x_k sequence a Markov process with initial measure $\mu_0 = h\,d\mu$.

Let M denote the maximal operator on $L_1(\Omega, P)$ defined by

$$MX(\omega) = \sup_{k \ge 1} |E\{X \,|\, x_k, x_{k+1}, \ldots\}(\omega)| \qquad (\omega \in \Omega, X \in L_1(\Omega, P))\,.$$

Then we have $\|MX\|_\infty \le \|X\|_\infty$ for all $X \in L_\infty(\Omega, P)$ and

$$P(\{MX > a\}) \le \frac{1}{a} \int\limits_{\{MX > a\}} |X|\,dP \qquad (a > 0, X \in L_1(\Omega, P))$$

(cf. e.g. [8], p. 69). Therefore Theorem 1 in [9] can be applied to infer that there exists a constant $B > 0$ such that

$$\int\limits_{\{MX > a\}} \frac{MX}{a}\,dP \le \int\limits_{\{B|X| > a\}} \frac{B|X|}{a} \left(\log \frac{B|X|}{a}\right) dP$$

for all $a > 0$ and $X \in R_1(\Omega, P)$, where we let

$$R_1(\Omega, P) = \left\{ X \in L_1(\Omega, P) : \int |X| \log^+ \frac{|X|}{a}\,dP < \infty \text{ for all } a > 0 \right\}.$$

(It is known (cf. [6]) that, since P is a finite measure, $R_1(\Omega, P)$ is a linear subspace of $L_1(\Omega, P)$, and $X \in R_1(\Omega, P)$ if and only if $\int |X| \log^+ |X| dP < \infty$.)

On the other hand, since $0 \le \widetilde{r}_t \le \widetilde{g}$ and $\widetilde{r}_t \downarrow 0$ by the definition of $\widetilde{r}_t$, and since $\widetilde{g}(x_0) \in R_1(\Omega, P)$ by (4), Lebesgue's convergence theorem can be applied to obtain

$$\lim_t \int_{\{M\tilde{r}_t(x_0)>a\}} \frac{1}{a} M\widetilde{r}_t(x_0)\, dP \le \lim_t \int_{\{B\tilde{r}_t(x_0)>a\}} \frac{1}{a} B\widetilde{r}_t(x_0) \left(\log \frac{B\widetilde{r}_t(x_0)}{a}\right) dP = 0$$

for all $a > 0$. Thus, immediately, $\lim_t \int M\widetilde{r}_t(x_0)dP = 0$. Since $t < s$ implies $M\widetilde{r}_t(x_0) > M\widetilde{r}_s(x_0) \ge 0$, it follows that

$$\lim_t E\{M\widetilde{r}_t(x_0) \mid x_0\} = 0 \quad \text{a.e. } (P)\,. \tag{8}$$

Further, since $r_t \in L_2$, it follows from Akcoglu's result [1] (see also [2]) that

$$\lim_n T_1^* \ldots T_n^* T_n \ldots T_1 r_t \quad \text{exists a.e. on } X\,. \tag{9}$$

Consequently,

$$\begin{aligned}
&\lim_N \sup_{n,m\ge N} |f_n - f_m| \\
&\quad \le \lim_N \sup_{n,m\ge N} |e_n - e_m| + \lim_N \sup_{n,m\ge N} |g_n - g_m| \\
&\quad \le 2\lim_N \sup_{n\ge N} |e_n| + \lim_N \sup_{n,m\ge N} |T_1^* \ldots T_n^* T_n \ldots T_1 r_t - T_1^* \ldots T_m^* T_m \ldots T_1 r_t| \\
&\qquad +2\lim_N \sup_{n\ge N} |T_1^* \ldots T_n^* T_n \ldots T_1(g - r_t)| \\
&\quad \le 2\varepsilon + 2\lim_N \sup_{n\ge N} \tau_1^* \ldots \tau_n^* \tau_n \ldots \tau_1(g - r_t) && \text{(by (2) and (9))} \\
&\quad \le 2\varepsilon + 2E\{M\widetilde{r}_t(x_0) \mid x_0\} && \text{(by (5), (6) and (7));}
\end{aligned}$$

and (8) shows that $(f_n(x))$, $n = 1, 2, \ldots$, is a Cauchy sequence for almost all x in X; thus $\lim_n f_n(x)$ exists almost everywhere, completing the proof.

REFERENCES

[1] M. A. Akcoglu, *Alternating sequences with nonpositive operators*, Proc. Amer. Math. Soc. 104 (1988), 1124–1130.

[2] M. A. Akcoglu and L. Sucheston, *Pointwise convergence of alternating sequences*, Canad. J. Math. 40 (1988), 610–632.

[3] I. Assani, *Alternating procedures in uniformly smooth Banach spaces*, Proc. Amer. Math. Soc. 104 (1988), 1131–1133.

[4] D. L. Burkholder, *Successive conditional expectations of an integrable function*, Ann. Math. Statist. 33 (1962), 887–893.

[5] J. L. Doob, *A ratio operator limit theorem*, Z. Wahrsch. Verw. Gebiete 1 (1963), 288–294.

[6] N. A. Fava, *Weak type inequalities for product operators*, Studia Math. 42 (1972), 271–288.

[7] U. Krengel, *Ergodic Theorems*, de Gruyter, Berlin 1985.
[8] J. Neveu, *Discrete-Parameter Martingales*, North-Holland, Amsterdam 1975.
[9] R. Sato, *Ergodic theorems for d-parameter semigroups of Dunford–Schwartz operators*, Math. J. Okayama Univ. 23 (1981), 41–57.
[10] N. Starr, *Operator limit theorems*, Trans. Amer. Math. Soc. 121 (1966), 90–115.

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