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CONTRACTIVE PROJECTIONS ON THE FIXED POINT SET OF L_∞ CONTRACTIONS

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It is well known that in Hilbert space any closed subspace is the range of a contractive projection, namely the orthogonal projection. Kakutani [Ka] showed that for a Banach space of dimension at least 3 the space is (isometrically) a Hilbert space if and only if each closed subspace admits a contractive projection. Lindenstrauss and Tzafriri [LT] showed that if each closed subspace of a Banach space is complemented then the space is equivalent to a Hilbert space.

Let X be a reflexive Banach space. Then if X has dimension at least 3 and is not a Hilbert space, there will be nonzero subspaces which admit no norm one projections. However, if T is a linear contraction on X, the mean ergodic theorem [Kr, p. 72] yields the fact that the fixed point set of T is the range of a contractive projection. Indeed, this theorem asserts that for each x in X,

$$Px = \lim_{N\to\infty} \frac{I + T + \ldots + T^N}{N+1} x$$

converges, defining a contractive projection onto $\mathrm{Fix}(T)$, the fixed point set of T, and, as P is obtained as a limit of functions of T , we also obtain $PT = TP$. Moreover, if Q is a projection (of any norm) onto $\mathrm{Fix}(T)$ which commutes with T then $Q = P$.

The result above was generalized by Lloyd in [L1]–[L3]. If $X = Y^*$ and $T = S^*$, where S is a linear contraction on Y, then there is a contractive projection P from X onto $\mathrm{Fix}(T)$ which commutes with T. His "construction" is the following: Let λ be a Banach limit. For x in X and y in Y define $\langle y, Px\rangle = \lambda\langle y, T^n x\rangle = \lambda\langle S^n y, x\rangle$. A slight variant of this construction is to replace T^n above by $(1/N_k)\sum_{n=1}^{N_k} T^n$ where $\{N_k\}$ is any fixed strictly increasing sequence of nonnegative integers. This alternate approach shows

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that if T is not weak-* mean ergodic, then P is not unique (but depends on $\{N_k\}$ at least).

Remark. The projection given by the mean ergodic theorem not only automatically commutes with T, it commutes with each operator which commutes with T. We cannot in general hope to construct a projection to have this stronger commutivity property. First note that if P_1 and P_2 are any two projections onto $\mathrm{Fix}(T)$ which commute with each other we have $P_1 = P_2$ trivially. Now if P_3 could be found with the stronger commutivity property and P_1 and P_2 are each projections onto $\mathrm{Fix}(T)$ which commute with T then P_3 would commute with both P_1 and P_2. So by the first observation $P_1 = P_3 = P_2$. Thus the existence of a doubly commuting projection would imply uniqueness of the T commuting projections. However, for the shift operator on ℓ_∞ (which can be taken to be of a finite measure) the space of constants is the fixed point set. There are many shift invariant projections onto this space.

In the case of a Markov operator T on L_∞ (where $T = S^*$ with S a *positive* contraction on L_1 satisfying $S^*1 \le 1$), the projection obtained by Lloyd's method is also *positive* (but it is not unique if, e.g., $S^*1 = 1$ and S has no nonzero fixed points).

Bruck [Br, p. 68] has given an example of a positive linear contraction T on $C[0,1]$ such that $\mathrm{Fix}(T)$ is not the range of any contractive projection (nor even the range of any nonlinear nonexpansive retraction).

The purpose of this paper is to study the problem for positive linear contractions in L_∞. We deal with a commutative semigroup $\mathcal{S}$ of contractions, and denote the set of common fixed points by $\mathrm{Fix}(\mathcal{S})$.

The important tools are:

1. The *binary ball intersection property*, enjoyed by L_∞: If B_α is a collection of closed balls such that any two balls intersect, then $\bigcap B_\alpha \neq \emptyset$. (This notion makes sense in any metric space; see [AP] for discussion).

2. Baillon's fixed point theorem for $\mathcal{S}$-invariant order intervals [B] (for a single contraction T see [Si] or [So]).

PROPOSITION 1. *Let $\mathcal{S}$ be an amenable semigroup of linear contractions in a dual Banach space $X = Y^*$.*

1. *There exists a linear contraction R such that $RT = R$ for every $T \in \mathcal{S}$, and $Rx = x$ for $x \in \mathrm{Fix}(\mathcal{S})$.*

2. *If every $T \in \mathcal{S}$ is the dual of an operator $\widehat{T}$ on Y, then R is a projection onto $\mathrm{Fix}(\mathcal{S})$ which commutes with $\mathcal{S}$.*

3. *If there exists a contractive projection onto $\mathrm{Fix}(\mathcal{S})$, then there exists one which commutes with $\mathcal{S}$.*

Proof. 1. Let λ be an invariant mean on $\mathcal{S}$. Define R by $\langle y, Rx\rangle = \lambda(\langle y, Tx\rangle)$ for $y \in Y$, $x \in X$. Clearly R is linear with $\|R\| \le 1$, $Rx = x$ for $x \in \mathrm{Fix}(\mathcal{S})$. By the invariance, $RT_0 = R$ for any $T_0 \in \mathcal{S}$.

2. For $T_0 \in \mathcal{S}$ we have $\langle y, T_0 R_x\rangle = \langle \widehat{T_0} y, Rx\rangle = \lambda(\langle \widehat{T_0} y, Tx\rangle) = \langle y, Rx\rangle$, by the invariance of λ. Hence $T_0 R = R$ for every $T_0 \in \mathcal{S}$. Hence $R^2 = R$, since $\langle y, R^2 x\rangle = \lambda(\langle y, TRx\rangle) = \lambda(\langle y, Rx\rangle) = \langle y, Rx\rangle$.

3. Let Q be a contractive projection onto $\mathrm{Fix}(\mathcal{S})$, and define $P = QR$. Then $TP = TQR = QR = P$ for $T \in \mathcal{S}$. Since $Rx = x$ for every $x \in \mathrm{Fix}(\mathcal{S})$, we have $RQ = Q$, hence $P^2 = QRQR = Q^2 R = QR = P$. Finally, $PT = QRT = QR = P$ for any $T \in \mathcal{S}$.

Remark. The first two parts are the abstract version of Lloyd's result. Part (3) was suggested by Yoav Benyamini, and yields a different and shorter proof than the authors' original geometric proof of the next result.

PROPOSITION 2. *Let $\mathcal{S}$ be a commutative semigroup of linear contractions on L_∞. Then there exists a contractive projection onto* $\mathrm{Fix}(\mathcal{S})$ *which commutes with $\mathcal{S}$.*

Proof. By Baillon's fixed point theorem, $\mathrm{Fix}(\mathcal{S})$ has the binary ball intersection property, hence it is a P_1-space [La, p. 93]. This shows that there is a linear contractive projection onto $\mathrm{Fix}(\mathcal{S})$, and the previous proposition finishes the proof.

Remark. The novelty in Proposition 2 is the fact that we can find a contractive projection which *commutes* with $\mathcal{S}$. The proof uses the fact that L_∞ is a *dual space*, and is valid for any (space linearly isometric to) $C(K)$ with K hyperstonian. The result for any K extremally disconnected compact Hausdorff space (i.e., for all P_1-spaces) is also true, but requires a different proof (which will appear elsewhere). Theorem 3 below will then apply to $C(K)$ for such K (with the same proof).

THEOREM 3. *Let $\mathcal{S}$ be a commutative semigroup of positive linear contractions on L_∞. Then there exists a positive contractive projection onto* $\mathrm{Fix}(\mathcal{S})$ *which commutes with $\mathcal{S}$.*

Proof. We will show that the projection obtained in Proposition 2 must be positive. In fact, any contractive projection P onto $\mathrm{Fix}(\mathcal{S})$ is positive.

Since $T0 = 0$, we can define

$$u = \bigvee \{\omega \in \mathrm{Fix}(\mathcal{S}) : 0 \le \omega \le 1\}\,.$$

For $T \in \mathcal{S}$ we have $Tu \ge u \ge 0$ and $T1 \le 1$, since T is positive. Hence the nonempty closed order interval $[u, 1]$ is $\mathcal{S}$-invariant, so by Baillon's theorem [B] must contain a fixed point $u' \in \mathrm{Fix}(\mathcal{S})$. But $u \le u' \le 1$, so by definition $u' \le u$ and u is a maximal $\mathcal{S}$-invariant function in $[0, 1]$. Moreover, if $\omega \in$

$\mathrm{Fix}(\mathcal{S})$ with $\|\omega\|_\infty \le 1$, then $|\omega| = |T\omega| \le T|\omega|$ for every $T \in \mathcal{S}$, so the $\mathcal{S}$-invariant order interval $[|\omega|, 1]$ contains a fixed point ω', hence $|\omega| \le \omega' \le u$.

Now $|2u-1| \le 1$, so applying the above to $\omega = P(2u-1)$ yields

$$2u - P1 = P(2u-1) \le u\,.$$

Hence $u \le P1 \le 1$, and therefore $P1 = u$, by the maximality of u.

Now for $0 \le f \le 1$ in L_∞ we have $|1-2f| \le 1$ and $|P(1-2f)| \le u$. Thus

$$u - 2Pf = P(1-2f) \le u$$

and $Pf \ge 0$. Hence P is positive. ∎

As noted in Proposition 2, $\mathrm{Fix}(\mathcal{S})$ has the binary ball intersection property. Hence $\mathrm{Fix}(\mathcal{S})$ is linearly isometric to $C(K)$, for K extremally disconnected [La, p. 93]. If $\mathcal{S}$ consists of positive operators, the isometry will be *positive*, as is shown below (by Theorem 3, we have to consider only $\mathcal{S} = \{P\}$ with P a positive contractive projection).

THEOREM 4. *Let $F \ne \{0\}$ be a closed subspace of L_∞. Then the following conditions are equivalent*:

(1) *F is the range of a positive contractive projection.*

(2) *In the induced order*, *F is a boundedly complete lattice with* (*strong*) *order unit.*

(3) *There exists an extremally disconnected compact Hausdorff space K with a positive linear isometry of $C(K)$ onto F.*

Proof. (1)⇒(2). Let P be a positive contractive projection with range F. By [LS], $F = \mathrm{Fix}(P)$ is a *boundedly complete* lattice in the induced order. Now u as defined in the proof of Theorem 3 (with $T = P$) is a maximal element in the unit ball of F, by its properties obtained in that proof.

(2)⇒(3). We show that F is an abstract M-space with unit. Let u be the order unit in F, and $f, g \in F$ nonnegative. Let $\alpha = \|f\|_\infty$, $\beta = \|g\|_\infty$. By maximality of u, $0 \le f \le \alpha u$, $0 \le g \le \beta u$. Denoting the lattice operations in F in the induced order by $\overset{F}{\vee}$, we have $f \overset{F}{\vee} g \le (\alpha \vee \beta)u$, so $\|f \overset{F}{\vee} g\| \le \|f\| \vee \|g\|$. The reverse inequality is obvious. Now by Kakutani's representation theorem F is isometrically and order isomorphic to $C(K)$ for some compact Hausdorff K. Since $C(K)$ is order complete (F is), K is extremally disconnected [La, p. 92].

(3)⇒(1). By the Nachbin–Kelley theorem, F is a P_1-space [La, p. 92], and there exists a contractive projection P from L_∞ onto F. We will show that P is positive. Let φ be a positive linear isometry of $C(K)$ onto F, and let $u = \varphi(1)$. Then u is maximal in the unit ball of F, and the positivity of P is proved by the arguments at the end of the proof of Theorem 3.

Remarks. 1. Our original proof, for (1)⇒(3), essentially constructed the representation. U. Krengel suggested to just reduce the problem to Kakutani's theorem.

2. The proofs use only the lattice properties of L_∞, but (unlike Theorem 3) they do not use the fact that it is a dual space. Hence L_∞ can be replaced in the theorem by $C(K')$, K' Stonian.

3. In the case $F = \mathrm{Fix}(T)$ for T a positive contraction this last result gives an abstract Feller boundary for T.

Proposition 2 and Theorem 3 have the following analogues for nonexpansive maps on L_∞.

PROPOSITION 5. *Let $\mathcal{S}$ be a commutative semigroup of nonexpansive maps on L_∞. If* $\mathrm{Fix}(\mathcal{S}) \neq \emptyset$, *then there exists a nonexpansive retract map onto* $\mathrm{Fix}(\mathcal{S})$ *which commutes with $\mathcal{S}$.*

Proof. By Baillon's theorem [B], if $\mathrm{Fix}(\mathcal{S})$ is nonempty, it has the binary ball intersection property. By [AP] there is a nonexpansive retract map Q of L_∞ onto $\mathrm{Fix}(\mathcal{S})$. Imitating the proof of part 1 of Proposition 1 we obtain a nonexpansive map R with the same properties as in Proposition 1. The computations in the proof of part 3 of Proposition 1 show that $P = QR$ is the desired retract map.

THEOREM 6. *Let $\mathcal{S}$ be a commutative semigroup of order preserving nonexpansive maps on L_∞. If* $\mathrm{Fix}(\mathcal{S}) \neq \emptyset$, *then there exists an order preserving nonexpansive retract map onto* $\mathrm{Fix}(\mathcal{S})$ *which commutes with $\mathcal{S}$.*

Proof. The case $\mathcal{S} = \{T^n\}$ was proved in [LS]. By using [B] instead of [Si] in the proof, we obtain Theorem 6.

Remarks. 1. The novelty in Proposition 5 (as in Proposition 2) is that the retract map *commutes* with $\mathcal{S}$. The proof of Proposition 5 uses the fact that L_∞ is a dual space, so it is not adaptable to general hyperconvex spaces.

2. The important feature in Theorems 3 and 6 is the order preserving property. Unlike the proof of Theorem 3, the proof of Theorem 6 constructs the retract map (as in [LS]) without using Proposition 5 (it uses [B], but not [AP]). Hence Theorem 6 is valid for $C(K)$ with K extremally disconnected compact Hausdorff space.

3. We give below a simple proof of the fixed point theorem of [B] for *order preserving* maps of a bounded interval into itself (nonexpansiveness is needed explicitly only in order to deduce that $\mathrm{Fix}(\mathcal{S})$ has the binary ball intersection property, which is used in the proof of Proposition 2. This is easily deduced from the fixed point theorem).

PROPOSITION 7. *Let $\mathcal{S}$ be a commutative semigroup of order preserving maps of a bounded order interval $I \subset L_\infty$ into itself. Then $\mathcal{S}$ has a fixed point in I.*

Proof. Let $\mathcal{T}$ be the set of $\mathcal{S}$-invariant subintervals of I, and order $\mathcal{T}$ by inclusion. If $\{I_\alpha\}$ is a decreasing chain in $\mathcal{T}$, then $\bigcap I_\alpha$ is an interval and is $\mathcal{S}$-invariant, so is the g.l.b. of $\{I_\alpha\}$. (If $I_\alpha = \{h : f_\alpha \le h \le g_\alpha\}$, $\bigvee f_\alpha$ and $\bigwedge g_\alpha$ are defined in the boundedly complete lattice L_∞, and $\bigcap I_\alpha = \{h : \bigvee_\alpha f_\alpha \le h \le \bigwedge_\alpha g_\alpha\}$). By Zorn's lemma $\mathcal{T}$ contains a minimal element $\bar{I} = \{h : \bar{f} \le h \le \bar{g}\}$ with $\bar{f}, \bar{g} \in I$.

Fix $T_0 \in \mathcal{S}$. By invariance of $\bar{I}, \bar{f} \le T_0\bar{f} \le \bar{g}$. For any $T \in \mathcal{S}$, we have $\bar{g} \ge T(T_0\bar{f}) = T_0(T\bar{f}) \ge T_0\bar{f}$ using commutativity. Hence $\{h : T_0\bar{f} \le h \le \bar{g}\}$ is $\mathcal{S}$-invariant, since all $T \in \mathcal{S}$ are order preserving. By minimality of $\bar{I}, T_0\bar{f} = \bar{f}$. Since $T_0 \in \mathcal{S}$ was arbitrary, $\bar{f}$ is a fixed point for $\mathcal{S}$.

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