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ON THE STRUCTURE OF $\mathfrak{G}$-SPACES [*]

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§1. Introduction. We study the class of those locally convex spaces which are reduced projective limits of Banach spaces with approximable linking maps, named $\mathfrak{G}$-spaces. Obviously each nuclear space is a $\mathfrak{G}$-space and each $\mathfrak{G}$-space is a Schwartz space with the Approximation Property. The first implication is strict, and it is an open question, posed by Ramanujan, whether the second is so or not. It was proved by Nelimarkka that each Fréchet–Schwartz space with the Bounded Approximation Property is a $\mathfrak{G}$-space. Therefore we find the following situation:

$$\begin{array}{lcl} \text{Fréchet–Schwartz} + \text{BAP} & \searrow & \\ & & \mathfrak{G}\text{-space} \to \text{Schwartz} + \text{AP} \\ \text{Fréchet nuclear} & \nearrow & \end{array}$$

(where the first two arrows cannot be reversed).

In this paper we introduce two "local" versions of the BAP: with respect to a finite number of seminorms (property G) and with respect to bounded sets (property L). These properties characterize the Schwartz G-spaces as precisely the $\mathfrak{G}$-spaces, and the co-Schwartz L-spaces as the spaces whose strong dual is a $\mathfrak{G}$-space. It follows that each Schwartz space with BAP is a $\mathfrak{G}$-space.

There is another sense in which it could be said that an lcs locally has BAP: when it possesses a fundamental system of neighborhoods of zero such that the associated Banach spaces have BAP. Let us call Schwartz spaces with this property $\mathfrak{G}^*$-spaces. The question of whether $\mathfrak{G}$- and $\mathfrak{G}^*$-spaces coincide arises, and extends a (somewhat different) question of Schottenloher. We prove that:

A Hausdorff lcs is a $\mathfrak{G}$-space if and only if it is a locally complemented subspace of a $\mathfrak{G}^*$-space

(the definition of local complementation is in the paper).

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Since Schwartz spaces with local BAP are $\mathfrak{G}$-spaces, and Schwartz spaces with local FDD are $\mathfrak{G}^*$-spaces, the above theorem generalizes (adding "local" before the key words) the well-known structure theorem of Benndorf: An lcs is a Schwartz space with BAP iff it is a complemented subspace of a Schwartz space with an FDD.

Therefore, this paper shows, in essence, that the class of $\mathfrak{G}$-spaces is an extension (obtained by localization) of the class of Schwartz spaces with BAP. This generalization has been shown to be a suitable framework for the study of approximation structures in Schwartz spaces, with the desirable bonus of including the nuclear spaces.

§ 2. Preliminaries. An operator means a linear continuous map. $\mathfrak{L}(E,F)$ denotes the class of all operators acting between the spaces E and F. $\mathfrak{F}$ and $\mathfrak{G}$ will represent the ideals of $\mathfrak{L}$ formed by finite rank and approximable operators respectively, that is, $\mathfrak{G}$ is the closure of $\mathfrak{F}$ in $\mathfrak{L}$ in the operator norm.

If E is a Hausdorff locally convex space (in short lcs), $U(E)$ denotes a fundamental system of absolutely convex closed neighborhoods of 0 in E. For $U \in U(E)$, with gauge p_U, E_U is the space $E/\operatorname{Ker} p_U$ endowed with the norm $\|\phi_U(x)\|_U = p_U(x)$, where ϕ_U is the quotient map. For $V \in U(E)$, $V \subset U$, the linking map T_{VU} is defined by $T_{VU}\phi_V(x) = \phi_U(x)$, and $\widehat{T}_{VU} \in \mathfrak{L}(\widehat{E}_V, \widehat{E}_U)$ denotes its extension to the completions. $B(E)$ denotes a fundamental system of absolutely convex closed bounded sets of E. When $A \in B(E)$, with gauge p_A, E_A is the space $\operatorname{span}(A)$ endowed with the topology of the norm p_A. When $B \in B(E)$, $A \subset B$, the operator i_{AB} is the canonical inclusion from E_A into E_B, and $\widehat{i}_{AB}$ its extension to the completions.

An lcs E is said to be a $\mathfrak{G}$-*space* (resp. a *Schwartz space*) when for each $U \in U(E)$ there exists a $V \in U(E)$ such that $\widehat{T}_{VU}$ is approximable (resp. is compact). E is said to be a *co*-$\mathfrak{G}$-*space* (resp. a *co-Schwartz space*) when for each $A \in B(E)$ there exists a $B \in B(E)$, $A \subset B$, such that $\widehat{i}_{AB}$ is approximable (resp. compact).

By using the tensorial representation of finite rank operators, it is easy to see that:

LEMMA 1. *Let X and Y be normed spaces and $T \in \mathfrak{G}(X,Y)$. Assume that Z is a dense subspace of Y with $T(X) \subset Z$. Then $T \in \mathfrak{G}(X,Z)$.*

An equivalent definition of $\mathfrak{G}$-space can now be given as follows: an lcs E is said to be a $\mathfrak{G}$-space when for each $U \in U(E)$ there exists $V \in U(E)$, $V \subset U$, such that $T_{VU} \in \mathfrak{G}(E_V, E_U)$.

An lcs E which is a subspace of an lcs F is said to be *complemented in* F when a continuous projection P from F onto E exists.

Definition 1. We will say that E is *locally complemented in* F when there exists a fundamental system of neighborhoods of 0, $U(F)$, such that for each $U \in U(F)$ $\widehat{E}_{U\cap E}$ is complemented in $\widehat{F}_U$.

The idea of local complementation is taken from [11, Lemma 14], which also proves that if E is complemented in F then it is locally complemented.

Let E and F be lcs and $T \in \mathfrak{L}(E,F)$. For each $V \in U(E)$ and $U \in U(E)$ such that $U \subset T^{-1}(V)$, T can be interpreted as an operator in $\mathfrak{L}(\widehat{E}_U, \widehat{E}_V)$. The induced map is defined by the equation $T(\phi_U x) = \phi_V(Tx)$, and then extended to the completions.

For general facts about the Approximation Property (AP) and the Bounded Approximation Property (BAP) we refer to [7], [9] and [8]. An lcs E has AP when for each precompact set K and each 0-nbhd $U \in U(E)$, a finite rank operator $T \in \mathfrak{F}(E)$ exists such that $(\mathrm{Id}_E - T)(K) \subset U$. E is said to have BAP if the identity on E belongs to the closure of an equicontinuous set of finite rank operators on E, the closure being taken in the topology of uniform convergence on precompact sets. When E is a separable Fréchet space this is equivalent to the existence of a sequence of finite rank operators pointwise convergent to the identity.

If E is an lcs and (A_n) a sequence of finite rank operators witnessing BAP in E, then the sequence (B_n) such that $B_1 = A_1$, and $B_n = A_n - A_{n-1}$ $(n \geq 2)$, is called a *partition of the identity* of E. When a partition (B_n) can be found satisfying not only $\sum_n B_n x = x$ uniformly over compact sets but also $B_n B_m = \delta_{nm} B_n$ then it is said to be a Finite-Dimensional Decomposition (FDD) of E.

§3. Internal structure. We will say that a net $(A_i)_{i\in I}$ of operators on E is *equicontinuous with respect to* $U \in U(E)$ when the set $\bigcap_i A_i^{-1}(U)$ is a 0-nbhd in E; and we will say that it is *equibounded with respect to* $B \in B(E)$ when $\bigcup_i A_i(B)$ is a bounded subset of E.

It is clear that $(A_i)_{i\in I}$ is equicontinuous with respect to U if and only if $(\phi_U A_i)_{i\in I}$ is equicontinuous; and equibounded with respect to B if and only if $(A_i i_B)_{i\in I}$ is equibounded.

Definition 2. A locally convex space E is said to be a *G-space* when for each $U \in U(E)$ there exists a net (A_i) of finite rank operators of $\mathfrak{F}(E)$, equicontinuous with respect to U, and such that for each $x \in E$ the net $(A_i x)$ converges to x in the seminorm p_U.

Theorem 1. *Let E be an lcs. E is a $\mathfrak{G}$-space if and only if E is a Schwartz G-space.*

Proof. Let E be a Schwartz G-space. Let (A_i) be a net of finite rank operators of E pointwise convergent to the identity with respect to p_U. By the equicontinuity with respect to U, taking $V \subset \bigcap_i A_i^{-1}(U)$, we may consider the net in $\mathfrak{L}(E_V, E_U)$, keeping the convergence feature. Since now $\|A_i\| \le 1$, we get the convergence of (A_i) to $\widehat{T}_{VU}$ uniformly on compact subsets of $\widehat{E}_V$. E being a Schwartz space, the net $(A_i\widehat{T}_{WV})$ is norm convergent to $\widehat{T}_{WU}$ for some $W \in U(E)$. Thus E is a $\mathfrak{G}$-space.

On the other hand, it is clear using the tensorial representation of finite rank operators and the isomorphism $(E_U)' = E'_U$ that a $\mathfrak{G}$-space is a G-space. ■

Definition 3. An lcs E is an *L-space* when for each $B \in B(E)$ there exists a net (A_i) of finite rank operators of $\mathfrak{F}(E)$, equibounded with respect to B, such that for each $x \in B$ the net $(A_i x)$ converges to x in the topology of E.

Theorem 2. *Let E be an lcs. The strong dual E'_b of E is a $\mathfrak{G}$-space if and only if E is a co-Schwartz L-space.*

Proof. Let E be a co-Schwartz space. By Theorem 1 we only need to prove that E'_b is a G-space. Let $A \in B(E)$, and find $B \in B(E)$ such that A is relatively compact in E_B; let (A_i) be a net of finite rank operators of $\mathfrak{F}(E)$ pointwise convergent to the identity on E_B. We use the equiboundedness of (A_i) to find a $C \in B(E)$ containing $A_i(B)$ for all $i \in I$; this yields that for any neighborhood V of 0 there is some scalar $k > 0$ such that $B \subset \bigcap_i A_i^{-1}(kV)$, which implies that $\|A_i\| \le k$ as operators in $\mathfrak{L}(E_B, E_V)$, and, therefore, that the convergence of (A_i) to the identity with respect to p_V is uniform on compact sets of E_B. We thus have

$$\sup_{x\in A} p_V(x - A_i x) \le n^{-1} \quad \text{for large } i\,.$$

Now, the net $(A'_i) \subset \mathfrak{F}(E'_b)$ satisfies

$$p_{A^\circ}(a - A'_i a) \le p_{V^\circ}(a) \sup_{x\in A} p_V(x - A_i x) \le p_{V^\circ}(a) n^{-1} \quad \text{for large } i\,,$$

where V is some neighborhood of 0. This gives the pointwise convergence of (A'_i) with respect to p_A.

We only need to prove the equicontinuity of (A'_i) with respect to A. Following the preceding reasoning we have

$$p_{A^\circ}(A'_i a) \le p_{C^\circ}(a) \sup_{x\in A} p_C(A_i x) \le 1 \quad \text{when } a \in C^\circ\,.$$

To prove the converse implication, assume that E'_b is a $\mathfrak{G}$-space or, equivalently (see [11]), E is a co-$\mathfrak{G}$-space. Given $A \in B(E)$ we can find a $B \in B(E)$ such that i_{AB} is a compact operator, and then a $C \in B(E)$

such that i_{BC} is an operator approximable by a sequence

$$A_n = \sum_{i=1}^{k(n)} a_i \otimes y_i$$

of finite rank operators, where $a_i \in (E_B)'$, $y_i \in E_C$ and where we assume that $p_C(x - A_n x) \le n^{-1} p_B(x)$.

Applying the lemma of [14] we will find functionals b_i belonging to the dual of E_B endowed with the induced topology, such that

$$|b_i(x) - a_i(x)| \le (nk(n)p_C(y_i))^{-1} \quad \text{when } x \in A\,.$$

We extend, by using the Hahn–Banach theorem, the b_i to functionals (with the same names) $b_i \in E'$, and form the operators

$$B_n = \sum_{i=1}^{k(n)} b_i \otimes y_i$$

which belong to $\mathfrak{F}(E)$. If $U \in U(E)$ then

$$p_U(x - B_n x) \le M p_C(x - B_n x) \le 2Mn^{-1} \quad \text{when } x \in A\,,$$

where M is a constant which depends on U and C (and thus on A). E is therefore an L-space. ■

Properties G and L can be regarded as generalizations of BAP, in some sense intermediate between BAP and AP: it is clear that in Banach spaces BAP, L and G are equivalent. In general, BAP implies both G and L, but the converse is not true (take a Fréchet nuclear space E without BAP (see [6]): E is clearly a $\mathfrak{G}$- and dual-$\mathfrak{G}$-space, and hence a G- and L-space). On the other hand, it can be proved, with the same techniques as for Theorems 1 and 2, that a G-space has AP and that an L-space whose compact sets form a Schwartz bornology (see [8]) also has AP.

It is an open question whether any Schwartz space with AP is necessarily a $\mathfrak{G}$-space. This was posed by Ramanujan in [12, problem 22]. Nelimarkka gave in [11] a partial answer: Fréchet–Schwartz spaces with BAP are $\mathfrak{G}$-spaces. Our Theorem 1 extends this result to general Schwartz spaces.

Remarks about UAP. The Uniform Approximation Property was defined in [3] as follows: a Hausdorff lcs E has UAP when for each $U \in U(E)$ there exist a $V \in U(E)$ and a sequence (T_n) of finite rank operators in $\mathfrak{F}(E)$ such that $p_U(x - T_n x) \le n^{-1} p_V(x)$ for all $x \in E$. It turns out that an lcs has UAP if and only if it is a $\mathfrak{G}$-space [3]. Analogously, we define co-UAP as follows: for each $A \in B(E)$ there exist a $B \in B(E)$ and a sequence (A_n) of finite rank operators in $\mathfrak{F}(E)$ such that $p_B(x - A_n x) \le n^{-1}$ for all $x \in A$. An lcs is a dual-$\mathfrak{G}$-space if and only if it has co-UAP [4]. These properties are relevant when studying approximation structures in Schwartz and co-Schwartz spaces [5].

§4. External structure. Besides Hilbertizable spaces, very little is known about the structure of the Banach spaces associated to a Schwartz space E: there is a fundamental system $U(E)$ of 0-neighborhoods such that for all $U \in U(E)$, E_U is a subspace of c_0; this is false for c_0 replaced with any other l_p, $1 < p < +\infty$.

We will treat this question for $\mathfrak{G}$-spaces. Theorem 1 suggests the following

DEFINITION 4. We call an lcs E a $\mathfrak{G}^*$-*space* when it is a Schwartz space and has a fundamental system of neighborhoods of zero whose associated Banach spaces have BAP.

The relation with $\mathfrak{G}$-spaces is given by the following:

THEOREM 3. *An lcs E is a $\mathfrak{G}$-space if and only if it is a locally complemented subspace of a $\mathfrak{G}^*$-space.*

Proof. The "if" part is clear. We will prove the "only if" part: let us assume that E is a Fréchet $\mathfrak{G}$-space. We shall give a rather detailed outline of the proof.

Using UAP it is possible to find a fundamental system of continuous seminorms $(p_k)_{k\in\mathbb{N}}$ in E, and sequences $(B_v^k)_{v\in\mathbb{N}}$ of finite rank operators of $\mathfrak{F}(E)$ satisfying:

1. $x = \sum_{v=1}^{\infty} B_v^k x$ (convergence in p_k),
2. $p_k(B_v^k x) \le x_v p_{k+1}(x)$, $x \in E$,

where (x_v) is a rapidly decreasing sequence with all terms positive. So we have for some constants C_n

$$\sum_{v=1}^{\infty} v^n p_k(B_v^k x) \le C_n p_{k+1}(x).$$

In this way the seminorms

$$q_{k,n}(x) = \sum_{v=1}^{\infty} v^n p_k(B_v^k x)$$

define the same topology in E as the (p_k), and we have $p_k(x) \le q_{k,n}(x) \le C_n p_{k+1}(x)$.

Let $(A_k)_{k\in\mathbb{N}}$ be a partition of $\mathbb{N}$ into an infinite number of infinite sets. Let $k : \mathbb{N} \to \mathbb{N}$ be the counting function on A_k : $k(v)$ = the vth member of A_k.

We define $F = \{(y_n) \in E^N : \ y_{k(v)} \in B_v^k(E),\ p_k(y_{k(v)})_{v\in\mathbb{N}} \in (s)$ for all $k \in \mathbb{N}$, and there exists $x^k \in E$ such that the series $\sum_{v=1}^{\infty} y_{k(v)}$ converges to x^k in the seminorm $p_k\}$.

For $Y = (y_n) \in F$ we put

$$s_{k,n}(Y) = \sum_{v=1}^{\infty} v^n p_k(y_{k(v)}), \quad t_{k,n}(Y) = \max\Big\{q_{k,n}\Big(\sum_{v=1}^{\infty} y_{(k+1)(v)}\Big), s_{k,n}(Y)\Big\}.$$

It is easy to see that both formulae define seminorms on F. A fundamental set of seminorms for a locally convex separated topology on F is given by

$$1_{h,m}(Y) = \max_{\substack{k\le h\\ n\le m}}\{s_{k,n}(Y)\}\,, \qquad 2_{h,m}(Y) = \max_{\substack{k\le h\\ n\le m}}\{t_{k,n}(Y)\}\,.$$

Both topologies coincide since $1_{k,n}(Y) \le 2_{k,n}(Y) \le C_n 1_{k+1,n}(Y)$.

We define $T : E \to F$ by $x \mapsto (x_n)$, where

$$x_{k(v)} = B_v^k(x)\,.$$

T is linear, injective and bi-continuous:

$$s_{k,n}(Tx) = \sum_{v=1}^{\infty} v^n p_k(B_v^k x) = q_{k,n}(x)\,,$$

thus $1_{h,m}(Tx)$ and $q_{h,m}(x)$ are equivalent (as well as $2_{h,m}(Tx)$ and $q_{h,m}(x)$).

We have proved that E is a subspace of F.

We shall now see that F is a $\mathfrak{G}$-space: Define $C_v^k : F \to F$ by $Y \mapsto (z_n)$, where

$$\begin{cases} z_n = y_{j(v)} & \text{when } n = j(v) \text{ and } j \le k;\\ z_n = 0 & \text{otherwise.}\end{cases}$$

Since $y_{k(v)} \in B_v^k(E)$, C_v^k is a finite rank operator. Moreover,

$$1_{h,m}\Big(\sum_{v=1}^{t} C_v^k(Y)\Big) = \max_{\substack{k\le h\\ n\le m}}\Big\{\sum_{v=1}^{t} v^n p_k(y_{k(v)})\Big\} \le 1_{h,m}(Y)\,,$$

which proves that for each $t \in \mathbb{N}$, $\sum_{v=1}^{t} C_v^k$ is a continuous operator in $\mathfrak{L}(F_{1_{h,m}}, F_{1_{h,m}})$ with $\|\sum_{v=1}^{t} C_v^k\| \le 1$.

To conclude that each associated Banach space $\widehat{F}_{1_{h,m}}$ has BAP, we only need to prove that $Y = \sum_{v=1}^{\infty} C_v^k(Y)$ (convergence in $1_{h,m}$). We will prove simultaneously that F is a $\mathfrak{G}$-space showing that it has UAP:

$$\begin{aligned}1_{h,m}\Big(Y - \sum_{v=1}^{t} C_v^k(Y)\Big) &= \max_{\substack{k\le h\\ n\le m}}\Big\{\sum_{v=t+1}^{\infty} v^n p_k(y_{k(v)})\Big\}\\ &\le t^{-1} \max_{\substack{k\le h\\ n\le m+1}}\Big\{\sum_{v=t+1}^{\infty} v^n p_k(y_{k(v)})\Big\} \le t^{-1} 1_{h,m+1}(Y)\,.\end{aligned}$$

Finally, we shall prove that E is locally complemented in F. We need the system $(2_{h,m})$: Define $P : (F, 2_{h,m}) \to (T(E), 2_{h,m})$ by $Y \mapsto P(Y) = (z_n)$, where

$$z_{k(v)} = B_v^k \Big(\sum_{v=1}^{\infty} y_{(h+1)(v)} \Big) .$$

It is easy to see that $P(Y) = T(\sum_{v=1}^{\infty} y_{(h+1)(v)})$. It is clear that P is linear. It is also continuous:

$$2_{h,m}(P(Y)) = 2_{h,m}\Big(T\Big(\sum_{v=1}^{\infty} y_{(h+1)(v)}\Big)\Big) = q_{h,m}\Big(\sum_{v=1}^{\infty} y_{(h+1)(v)}\Big) \le 2_{h,m}(Y) .$$

Since $P(Tx) = T(\sum_{v=1}^{\infty} B_v^{h+1} x) = Tx$ the restriction of P to $T(E)$ coincides with the identity on $T(E)$. ∎

When E is not metrizable, the same proof applies with only minor changes: instead of a partition of $\mathbb{N}$ we need a partition of a set I with the cardinality of a base of 0-nbhds of E.

Remarks. The system $(2_{h,m})$ in F is needed in order to obtain the local complementation of E in F; the system $(1_{h,m})$ is needed in order to obtain associated Banach spaces with BAP. Note that those Banach spaces have even an FDD. In fact, the constructed space F does have a local FDD (an obvious definition in the same spirit as that of local BAP).

Following [6, pp. 169–170] it can be proved that:

PROPOSITION 1. *If E is a $\mathfrak{G}$-space with a local FDD, then it is a $\mathfrak{G}^*$-space.*

Having Theorems 1 and 3 we see that the class of $\mathfrak{G}$-spaces can be regarded as a generalization of the class of Schwartz spaces with BAP: in Theorem 1 it is proved that the $\mathfrak{G}$-spaces are the Schwartz spaces with local BAP. We look now to [2, Thm. 3]: "E is a Fréchet–Schwartz space with BAP if and only if it is a complemented subspace of a Fréchet–Schwartz space with an FDD". The passage to "local" allows us to forget the metrizability condition to obtain: "E is a Schwartz space with local BAP if and only if it is a locally complemented subspace of a Schwartz space F with a local FDD". Theorem 1 asserts that such an E is exactly a $\mathfrak{G}$-space. Proposition 1 says that F is a $\mathfrak{G}^*$-space, and thus we obtain the statement of Theorem 3.

Remark about the origin of the embedding problem. An lcs E is said to be a *DFC-space* if $E = F'_c$ for some Fréchet space F (here F'_c represents the dual space endowed with the topology of uniform convergence on compact subsets of F).

The question of Schottenloher cited in the introduction (see [13]) is:

"Does every DFC-space E with AP have a fundamental system of neighborhoods of zero (U_i) such that the associated Banach spaces $\widehat{E}_i$ have BAP?"

First we note that:

PROPOSITION 2 [4, 2.2.5]. *Let F be a Fréchet–Montel space. Then the following assertions are equivalent*: 1. *F has AP.* 2. *F is an L-space.* 3. *F'_b is a G-space.* 4. *F'_b is a $\mathfrak{G}$-space.* 5. *F'_b has AP.*

The proof follows from the techniques of Theorems 1 and 2. The equivalence of 1, 3 and 5 can also be found in [11]. From the proof it can be clearly seen that we can drop the assumption "Montel" on F, thus obtaining the results for F'_c instead of F'_b.

Therefore the problem of Schottenloher can be viewed as the problem:

"Is each $\mathfrak{G}$-space a $\mathfrak{G}^*$-space ?"

for some special $\mathfrak{G}$-spaces.

In the first part of [10] M. L. Lourenço proves:

"If E is a DFC-space with AP, then E is a compact projective limit of a family of Banach spaces with a monotone Schauder basis".

This is tantamount to saying that special $\mathfrak{G}$-spaces (DFC-spaces with AP) are subspaces of special $\mathfrak{G}^*$-spaces (with a local monotone basis). This result of Lourenço gives, for those particular $\mathfrak{G}$-spaces, better information about the "big" space, while our Theorem 3 gives a better knowledge of the quality of the embedding. I believe that it is not possible to obtain both at a time: a local complementation embedding in a bigger space with a local Schauder basis. See [4] for some related problems and additional information.

Whether each $\mathfrak{G}$-space is a $\mathfrak{G}^*$-space remains an unsolved question. On the other hand, the universal Schwartz space $[l_\infty, \mu(l_\infty, l_1)]$ is a $\mathfrak{G}^*$-space, and thus each Schwartz space is a subspace of a $\mathfrak{G}^*$-space. We can prove still more:

PROPOSITION 3. *Each Schwartz space is a subspace of a $\mathfrak{G}^*$-space with BAP.*

Proof. We use a result of Bellenot [1] which asserts that each Fréchet–Schwartz space is a subspace of a $\mathfrak{G}^*$-space with BAP (in combination with Theorem 1 and Proposition 1).

It is not hard to prove that each Schwartz space is a subspace of a product of Fréchet–Schwartz spaces. Since the product of $\mathfrak{G}^*$-spaces is again a $\mathfrak{G}^*$-space, the proof is complete. ∎

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REFERENCES

[1] S. F. Bellenot, *Each Schwartz Fréchet space is a subspace of a Schwartz Fréchet space with an unconditional basis*, Compositio Math. 42 (1981), 273–278.
[2] A. Benndorf, *On the relation of the bounded approximation property and a finite dimensional decomposition in nuclear Fréchet spaces*, Studia Math. 75 (1983), 103–119.
[3] J. M. F. Castillo, *An internal characterization of G-spaces*, Portugal. Math. 44 (1987), 63–67.
[4] —, *La estructura de los $\mathfrak{G}$-espacios*, Tesis Doctoral, Publ. Dept. Mat. Univ. Extremadura 16, 1986.
[5] —, *On the BAP in Fréchet Schwartz spaces and their duals*, Monatsh. Math. 105 (1988), 43–46.
[6] E. Dubinsky, *The Structure of Nuclear Fréchet Spaces*, Lecture Notes in Math. 720, Springer, 1979.
[7] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955).
[8] H. Jarchow, *Locally Convex Spaces*, B. G. Teubner, Stuttgart 1981.
[9] G. Köthe, *Topological Vector Spaces I, II*, Springer, 1969, 1979.
[10] M. L. Lourenço, *A projective limit representation of DFC-spaces with the approximation property*, J. Math. Anal. Appl. 115 (1986), 422–433.
[11] E. Nelimarkka, *The approximation property and locally convex spaces defined by the ideal of approximable operators*, Math. Nachr. 107 (1982), 349–356.
[12] S. Rolewicz, *On operator theory and control theory*, in: Proc. Internat. Conf. on Operator Algebras, Ideals, and their Applications in Theoretical Physics, Leipzig 1977, Teubner Texte zur Math., Teubner, 1978, 114–118.
[13] M. Schottenloher, *Cartan–Thullen theorem for domains spread over $\mathcal{DFM}$-spaces*, J. Reine Angew. Math. 345 (1983), 201–220.
[14] T. Terzioğlu, *Approximation property of co-nuclear spaces*, Math. Ann. 191 (1971), 35–37.

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