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A LOCAL ALGEBRA STRUCTURE FOR H^p OF THE POLYDISC

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The Hardy space $H^p(\Delta)$ for $0 < p < \infty$ is the set of all holomorphic functions on the open unit disk Δ for which the norm

$$\|f\|_{H^p} = \sup_{r<1}\left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta\right)^{1/p}$$

is finite. Wigley [7] showed that, for $p \geq 1$, this Banach space can be given a Banach algebra structure under the Duhamel product (we use "$*$" to represent the product, although it is not the usual convolution of two functions):

$$(1)\qquad f * g(z) = \frac{d}{dz}\int_0^z f(z-t)g(t)\, dt = \int_0^z f(z-t)g'(t)\, dt + f(z)g(0)\,.$$

This is by no means the only natural multiplication that this vector space could be given. The usual convolution of the boundary values of f on the set $\mathbb{T}$ of complex numbers of absolute value 1 makes $H^p(\Delta)$ into a semisimple Banach algebra when $p \geq 1$ [5].

In [6] the authors note that the Banach algebra $H^p(\Delta)$ with the Duhamel product has a singly generated Schauder basis. The question is posed whether $H^p(\Delta^n)$ can be endowed with a Banach algebra structure for which the natural basis is finitely generated. In this note we show that this is the case, and we extend the result to the non-Banach space case of $p < 1$. Moreover, the product we define on $H^p(\Delta^n)$ is a natural extension of the Duhamel product on $H^p(\Delta)$. This result follows from a more general argument on vector-valued analytic functions. We show that for an appropriate topological algebra $\mathbf{B}$ the space $H^p(\Delta, \mathbf{B})$ of $\mathbf{B}$-valued analytic functions for which the "usual" H^p norm is finite is a topological algebra under the Duhamel product, and the radical of $H^p(\Delta, \mathbf{B})$ is naturally isomorphic to

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the radical of $\mathbf{B}$. The main tool in our proof is the nontangential maximal function for $\mathbf{B}$-valued analytic functions. Our proofs simplify those given in [7] for the case $\mathbf{B} = \mathbb{C}$, and extend them to the case $p < 1$.

1. Main results. Let $\mathbf{B}$ be a complex topological vector space which is a complete metric space such that the dual of $\mathbf{B}$ separates points of $\mathbf{B}$, and let Ω be a region in $\mathbb{C}$. Then $f : \Omega \to \mathbf{B}$ is *analytic* if for every x^* in the dual of $\mathbf{B}$, the function $x^*(f(z))$ is an analytic function from Ω to $\mathbb{C}$. It can then be shown that f is continuous in the metric topology of $\mathbf{B}$ and that the usual line integrals one needs for the basic theorems of complex function theory can be treated as Riemann integrals, with the Riemann sums converging in the metric topology of $\mathbf{B}$ [3].

Fix p such that $0 < p \le \infty$. If $p \ge 1$, let $\mathbf{B}$ be a Banach space. If $0 < p < 1$, let $\mathbf{B}$ be a "p-normed F-space": that is, $\mathbf{B}$ has a homogeneous form $\|\cdot\|_{\mathbf{B}}$ such that $\|\cdot\|_{\mathbf{B}}^p$ satisfies the triangle inequality and gives a complete metric on the space (see [8]). In this case, we also suppose that the dual of $\mathbf{B}$ separates points of $\mathbf{B}$. We define $H^p(\varDelta, \mathbf{B})$ to be the (Banach or F-) space of $\mathbf{B}$-valued analytic functions on $\varDelta$ for which the "usual" H^p norm is finite:

$$|||f|||_{H^p} = \sup_{r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|_{\mathbf{B}}^p \, d\theta \right)^{1/p}.$$

Now suppose $\mathbf{B}$ is actually a (Banach or F-) algebra with identity e, end let $*_{\mathbf{B}}$ denote the algebra product on $\mathbf{B}$. (A large part of the Gelfand theory for commutative complex Banach algebras holds for such algebras, as is shown in [8].) We define the *Duhamel product* on $H^p(\varDelta, \mathbf{B})$ by

$$f * g(z) = \frac{d}{dz} \int_0^z f(z-t) *_{\mathbf{B}} g(t) \, dt \,. \tag{2}$$

Our main theorems are these:

THEOREM 1. *For $0 < p \le \infty$, the product defined above is a bounded bilinear form on $H^p(\varDelta, \mathbf{B})$ and hence $H^p(\varDelta, \mathbf{B})$ is a Banach algebra (for $p \ge 1$) or an F-algebra (for $p < 1$).*

THEOREM 2. *For $0 < p < \infty$, there is a natural isomorphism between the maximal ideal space of $H^p(\varDelta, \mathbf{B})$ and that of $\mathbf{B}$.*

The Hardy space of the polydisc, $H^p(\varDelta^n)$, is defined as those functions analytic on $\varDelta \times \ldots \times \varDelta$ for which the following norm is finite:

$$\|f\|_{H^p} = \sup_{r_1<1} \ldots \sup_{r_n<1} \left(\frac{1}{(2\pi)^n} \int_0^{2\pi} \ldots \int_0^{2\pi} |f(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n})|^p \, d\theta_1 \ldots d\theta_n \right)^{1/p}.$$

If $p \geq 1$, this is a Banach space. If $0 < p < 1$, this is a p-normed F-space, and it is known that its dual separates points, as is shown in [2] and [4]. The product we define on this space is the following:

$$
\begin{aligned}
(3) \quad & f * g(z_1, \ldots, z_n) \\
& = \frac{\partial^n}{\partial z_1 \ldots \partial z_n} \int_0^{z_n} \ldots \int_0^{z_1} f(z_1 - t_1, \ldots, z_n - t_n) g(t_1, \ldots, t_n)\, dt_1 \ldots dt_n \,.
\end{aligned}
$$

If $H^p(\Delta^{n-1})$ is an algebra with the product (3), then by Theorem 1, $H^p(\Delta, H^p(\Delta^{n-1}))$ is an algebra. By Wigley's result [7], or by taking $\mathbf{B} = \mathbb{C}$ in our Theorem 1, $H^p(\Delta)$ is an algebra with the product (1), so an induction yields that $H^p(\Delta, H^p(\Delta^{n-1}))$ is an algebra for all $n \geq 1$. (We will take $H^p(\Delta^n) = \mathbb{C}$ when $n = 0$.) Moreover, since $H^p(\Delta)$ is a *local* algebra ([7] or Theorem 2 with $\mathbf{B} = \mathbb{C}$), Theorem 2 and an induction show that $H^p(\Delta, H^p(\Delta^{n-1}))$ is also a *local* algebra for all $n \geq 1$. We show in Section 5 that the algebras $H^p(\Delta, H^p(\Delta^{n-1}))$ and $H^p(\Delta^n)$ are naturally isomorphic. Thus we have:

COROLLARY 3. $H^p(\Delta^n)$, *for* $0 < p \leq \infty$, *with the product given in* (3), *is a local* (*Banach or F*-) *algebra.*

For $p = \infty$, a separate argument is required, but an argument following the lines of Wigley's paper—embedding H^∞ into the algebra of formal power series in n variables—works here as well.

2. Hardy spaces of vector-valued functions. **B**-valued H^p theory includes certain analogues of results known in the numerically-valued case, which we present without proof.

An important and well-known [1, p. 36] estimate of the size of an H^p function is

$$|f(z)| \leq \left(\frac{1 + |z|}{1 - |z|}\right)^{1/p} \|f\|_{H^p} \,.$$

This estimate is true for all p, $0 < p \leq \infty$. Its proof follows from the fact that $|f(z)|^p$ is a subharmonic function of z for all holomorphic f and for all positive p. We write this simply as

$$|f(z)| \leq C_p (1 - |z|)^{-1/p} \|f\|_{H^p}$$

and an application of Cauchy's estimates gives us its close relative,

$$|f'(z)| \leq C_p (1 - |z|)^{-(1+1/p)} \|f\|_{H^p} \,.$$

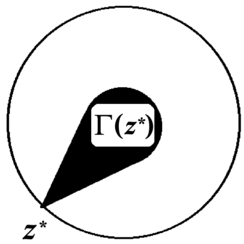
(C_p refers to any constant which depends only on p. We use the same symbol to represent constants that are not necessarily the same from usage to usage.)

For $\mathbf{B}$-valued holomorphic functions f, $\|f(z)\|_{\mathbf{B}}^p$ is also subharmonic, hence the analogues of these estimates hold:

$$\|f(z)\|_{\mathbf{B}} \le C_p(1-|z|)^{-1/p}|||f|||_{H^p}\,, \tag{4}$$

$$\|f'(z)\|_{\mathbf{B}} \le C_p(1-|z|)^{-(1+1/p)}|||f|||_{H^p}\,. \tag{5}$$

We will also need the *nontangential maximal function* of f. We define this as follows: For each point z^* on the boundary of Δ, we construct the "cone" or "nontangential approach region" based at z^* as shown in the figure (see [9]).



The nontangential approach region

The nontangential maximal function of f at z^* is defined as

$$Nf(z^*) = \sup_{z\in\Gamma(z^*)} |f(z)|\,.$$

(Of course, we can do the same with $\mathbf{B}$-valued functions, using the norm instead of the absolute value. In any case, Nf is itself real-valued.) The following fact is well known (see [9]): $f\in H^p(\Delta)$ if and only if $Nf\in L^p(\mathbb{T})$. In particular, we have the following (that it holds, even in the $\mathbf{B}$-valued case, involves once again noting that $\|f(z)\|_{\mathbf{B}}^p$ is subharmonic):

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |Nf(e^{i\theta})|^p\,d\theta\right)^{1/p} \le C_p|||f|||_{H^p}\,. \tag{6}$$

3. Proof of Theorem 1. Splitting the integral for $f*g$ into two pieces and integrating one of them by parts, we have

$$\begin{aligned} f*g(z) &= \int_0^{z/2} f(z-t) *_{\mathbf{B}} g'(t)\,dt + \int_{z/2}^{z} f(z-t) *_{\mathbf{B}} g'(t)\,dt + f(z) *_{\mathbf{B}} g(0) \\ &= \int_0^{z/2} f(z-t) *_{\mathbf{B}} g'(t)\,dt + \int_{z/2}^{z} f'(z-t) *_{\mathbf{B}} g(t)\,dt \\ &\quad + f(z) *_{\mathbf{B}} g(0) + f(0) *_{\mathbf{B}} g(z) - f(z/2) *_{\mathbf{B}} g(z/2) \end{aligned}$$

$$= \int_0^{z/2} f(z-t) *_{\mathbf{B}} g'(t) + g(z-t) *_{\mathbf{B}} f'(t)\, dt$$
$$+ f(z) *_{\mathbf{B}} g(0) + f(0) *_{\mathbf{B}} g(z) - f(z/2) *_{\mathbf{B}} g(z/2)\,.$$

Now apply inequality (5) to the derivatives under the integral and inequality (4) to the boundary term. This, along with the fact that $|t| \le \frac{1}{2}|z| \le \frac{1}{2}$, gives

$$\|f * g(z)\|_{\mathbf{B}} \le C_p\Big[|||g|||_{H^p} \int_0^{z/2} \|f(z-t)\|_{\mathbf{B}}\, dt + |||f|||_{H^p} \int_0^{z/2} \|g(z-t)\|_{\mathbf{B}}\, dt$$
$$+ |||g|||_{H^p} f(z) + |||f|||_{H^p} g(z) + |||f|||_{H^p} |||g|||_{H^p}\Big]\,.$$

However, if $z = re^{i\theta}$, then $z \in \Gamma(e^{i\theta})$ and thus $\|f(z)\|_{\mathbf{B}} \le Nf(e^{i\theta})$ and $\|g(z)\|_{\mathbf{B}} \le Ng(e^{i\theta})$. Thus at each point $z = re^{i\theta}$,

$$\|f * g(re^{i\theta})\|_{\mathbf{B}} \le C_p[|||g|||_{H^p} Nf(e^{i\theta}) + |||f|||_{H^p} Ng(e^{i\theta}) + |||f|||_{H^p} |||g|||_{H^p}]\,.$$

Raising this to the pth power and integrating around the circle yields

$$|||f * g(z)|||_{H^p} \le C_p |||f|||_{H^p} |||g|||_{H^p}\,.$$

4. Proof of Theorem 2. Since polynomials are dense in $H^p(\Delta, \mathbf{B})$ for $0 < p < \infty$, the product is completely determined by its action on the powers of z. This action can be summarized by the following lemma, whose proof is a straightforward calculation:

LEMMA 4. $a_n z^n * a_m z^m = \dfrac{n!m!}{(n+m)!}(a_n *_{\mathbf{B}} a_m) z^{n+m}$.

We define Z^{*n} to be the n-fold $*$-product of Z with itself, where Z denotes the function in $H^p(\Delta, \mathbf{B})$ whose value at point z is z times the identity element of $\mathbf{B}$. It follows from Lemma 4 that $(e\cdot z)^n = n! Z^{*n}$. Thus, if an element $f \in H^p(\Delta, \mathbf{B})$ can be written as a convergent series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for coefficients $a_n \in \mathbf{B}$, then it can be written in terms of the algebra as

$$f = \sum_{n=0}^{\infty} A_n Z^{*n}$$

where $A_n = n! a_n$. If we also have $g = \sum_{n=0}^{\infty} B_n Z^{*n}$, then

$$(7) \quad f * g = \Big(\sum_{n=0}^{\infty} A_n Z^{*n}\Big) * \Big(\sum_{n=0}^{\infty} B_n Z^{*n}\Big) = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} A_{n-k} *_{\mathbf{B}} B_k\Big) Z^{*n}\,.$$

In particular, for constant functions $f = A_0 \cdot \mathit{1}$, $g = B_0 \cdot \mathit{1}$ in $H^p(\Delta, \mathbf{B})$, $f * g = (A_0 *_{\mathbf{B}} B_0) \cdot \mathit{1}$.

Again, if the appropriate power series converge in $H^p(\varDelta, \mathbf{B})$, and if ϕ is a bounded multiplicative linear functional on $H^p(\varDelta, \mathbf{B})$, then

$$\phi(f) = \sum_{n=0}^{\infty} n!\phi(a_n Z^{*n}) = \sum_{n=0}^{\infty} n!\phi(a_n \cdot \mathit{1})[\phi(Z)]^n .$$

Since ϕ is bounded, this last series must converge for all $f \in H^p(\varDelta, \mathbf{B})$.

Consider as a particular case $f(z) = \sum_{n=0}^{\infty}(e/n^2)z^n$, where e is the identity element of $\mathbf{B}$. By direct calculation, this $f \in H^p(\varDelta, \mathbf{B})$ for all p, since it actually defines a continuous function on the closure of $\varDelta$, and the power series converges in the topology of $H^p(\varDelta, \mathbf{B})$. Then

$$\phi(f) = \phi(e \cdot \mathit{1}) \sum_{n=0}^{\infty} \frac{n!}{n^2}[\phi(Z)]^n .$$

So either $\phi(e \cdot \mathit{1}) = 0$, in which case ϕ is the zero functional, or the sum must converge; but the only way for the sum to converge is for $\phi(Z)$ to be 0. Thus for every polynomial $f = \sum_{n=0}^{N} A_n Z^{*n}$ in $H^p(\varDelta, \mathbf{B})$, $\phi(f) = \phi(A_0 \cdot \mathit{1})$. Since such polynomials are dense in $H^p(\varDelta, \mathbf{B})$ for $0 < p < \infty$, it follows from (7) that ϕ must be a multiplicative linear functional on the subalgebra of $H^p(\varDelta, \mathbf{B})$ consisting of the elements of the form $a \cdot \mathit{1}$, $a \in \mathbf{B}$. But this subalgebra is simply an isomorphic copy of $\mathbf{B}$, so that $\psi : \mathbf{B} \to \mathbb{C}$ given by $\psi(a) = \phi(a \cdot \mathit{1})$ defines a multiplicative linear functional on $\mathbf{B}$, and the correspondence $\phi \leftrightarrow \psi$ is a homeomorphism of the maximal ideal spaces of $H^p(\varDelta, \mathbf{B})$ and $\mathbf{B}$.

5. Proof of Corollary 3. The space $H^p(\varDelta, H^p(\varDelta^{n-1}))$ can be identified with $H^p(\varDelta^n)$. To do this, for each $f(z_1, \ldots, z_n) \in H^p(\varDelta^n)$, define $F : \varDelta \to H^p(\varDelta^{n-1})$ by letting $F(z_n)$ be that function of $z_1, \ldots, z_{n-1}$ whose values are given by $F(z_n)(z_1, \ldots, z_{n-1}) = f(z_1, \ldots, z_n)$. Now let us compute the norm of F:

$$\begin{aligned} |||F|||_{H^p(\varDelta, H^p(\varDelta^{n-1}))} &= \sup_{r_n<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|F(r_n e^{i\theta_n})\|^p_{H^p(\varDelta^{n-1})}\, d\theta_n \right)^{1/p} \\ &= \sup_{r_1<1} \ldots \sup_{r_n<1} \left(\frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} |f(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n})|^p \, d\theta_1 \ldots d\theta_n \right)^{1/p} \\ &= \|f\|_{H^p(\varDelta^n)} . \end{aligned}$$

Thus, this map is an isometry of the two spaces.

Finally, a direct calculation shows that the product (3) on $H^p(\varDelta, H^p(\varDelta^{n-1}))$ agrees with the product (2) on $H^p(\varDelta^n)$ under the correspondence of the previous paragraph. Therefore these spaces are isomorphic as Banach *algebras*.

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