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ON C-SETS AND PRODUCTS OF IDEALS

BY

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Let X, Y be uncountable Polish spaces and let μ be a complete σ-finite Borel measure on X. Denote by $\mathcal{K}$ and $\mathcal{L}$ the families of all meager subsets of X and of all subsets of Y with μ measure zero, respectively. It is shown that the product of the ideals $\mathcal{K}$ and $\mathcal{L}$ restricted to C-sets of Selivanovskiĭ is σ-saturated, which extends Gavalec's results.

0. Preliminaries. For a set X, let $\mathcal{P}(X)$ and $\mathcal{P}^{<\omega}(X)$ denote the families of all subsets of X and of all finite subsets of X, respectively.

For $A \subset X_1 \times X_2$ and $x_1 \in X_1$, we set $A_{x_1} = \{x_2 \in X_2 : (x_1, x_2) \in A\}$. Consider σ-fields $\mathcal{S}_1 \subset \mathcal{P}(X_1)$, $\mathcal{S}_2 \subset \mathcal{P}(X_2)$, $\mathcal{S} \subset \mathcal{P}(X_1 \times X_2)$ and ideals $\mathcal{I}_1 \subset \mathcal{S}_1$, $\mathcal{I}_2 \subset \mathcal{S}_2$. Set $A(\mathcal{I}_2) = \{x_1 \in X_1 : A_{x_1} \notin \mathcal{I}_2\}$. Assuming that

$$(*) \qquad A_{x_1} \in \mathcal{S}_2 \quad \text{for all } A \in \mathcal{S} \text{ and } x_1 \in X_1\,,$$

$$(**) \qquad A(\mathcal{I}_2) \in \mathcal{S}_1 \quad \text{for all } A \in \mathcal{S}\,,$$

let us define

$$(***) \qquad \mathcal{I}_1 \times \mathcal{I}_2 = \{A \in \mathcal{S} : A(\mathcal{I}_2) \in \mathcal{I}_1\}\,.$$

This is an ideal in $\mathcal{S}$ called the *product* of $\mathcal{I}_1$ and $\mathcal{I}_2$ (cf. [5], [4], [7]).

For a family $\mathcal{F} \subset \mathcal{P}(X)$ and an ideal $\mathcal{I} \subset \mathcal{P}(X)$ define

$$\mathcal{I}|\mathcal{F} = \{A \subset X : A \subset B \text{ for some } B \in \mathcal{I} \cap \mathcal{F}\} \quad \text{(cf. [4])}\,.$$

We say that an ideal $\mathcal{I}$ contained in a field $\mathcal{S}$ is *σ-saturated* in $\mathcal{S}$ if, whenever $\mathcal{F} \subset \mathcal{S} \setminus \mathcal{I}$ is uncountable, there are distinct $A, B \in \mathcal{F}$ such that $A \cap B \notin \mathcal{I}$. If $\mathcal{I}$ is a σ-ideal and $\mathcal{S}$ is a σ-field, then $\mathcal{I}$ is σ-saturated if there is no uncountable family $\mathcal{F} \subset \mathcal{S} \setminus \mathcal{I}$ of disjoint sets.

We shall consider the σ-fields $\mathcal{B}(X)$ and $\mathcal{C}(X)$ of all Borel sets and of all C-sets of Selivanovskiĭ (cf. [9]), respectively, in a given uncountable Polish space X. Recall that $\mathcal{C}(X)$ is the smallest class containing the open sets which is closed under complementation and Suslin's operation (A); it forms a σ-field properly containing $\mathcal{B}(X)$ (cf. [2]) and can be described by the following hierarchy (cf. [2], [10]):

$\mathcal{A}_0(X) = \mathcal{C}_0(X) = \mathcal{B}(X)$,
$\mathcal{A}_\beta(X) =$ the sets obtainable by (A) from indexed systems of elements of $\bigcup_{\alpha<\beta} \mathcal{C}_\alpha(X)$ (for $0 < \beta < \omega_1$),
$\mathcal{C}_\beta(X) =$ the σ-field generated by $\mathcal{A}_\beta(X)$,
$\mathcal{C}(X) = \bigcup_{\alpha<\omega_1} \mathcal{C}_\alpha(X)$.

By induction through that hierarchy one can easily prove

(0.1) PROPOSITION. *If X_1, X_2 are Polish spaces, then for any $x_1 \in X_1$ and $A \in \mathcal{C}(X_1 \times X_2)$ we have $A_{x_1} \in \mathcal{C}(X_2)$.*

Consequently, condition $(*)$ is fulfilled by $\mathcal{S}_2{=}\mathcal{C}(X_2)$ and $\mathcal{S}{=}\mathcal{C}(X_1{\times}X_2)$.

In [5] Gavalec described general assumptions under which condition $(**)$ holds if ideals $\mathcal{I}_1 \subset \mathcal{B}(X_1)$, $\mathcal{I}_2 \subset \mathcal{B}(X_2)$ are given and X_1, X_2 denote topological spaces; moreover, he proved that then the ideal $\mathcal{I}_1 \times \mathcal{I}_2 \subset \mathcal{B}(X_1 \times X_2)$ is σ-saturated if $\mathcal{I}_1$, $\mathcal{I}_2$ are σ-saturated. In particular, it turns out that the mixed measure-category product of ideals (i.e. the product of the ideal of all meager sets and the ideal of all sets having measure zero) restricted to the plane Borel sets is σ-saturated. A simple extension of that result, however still for the Borel case, can be found in [1]. Gavalec's method is based on quasi-maximal systems of quasi-ideals arising from certain properties of linear sets concerning measure and category. In this paper we adopt the ideals of [5] to the case where the fields of Borel sets are replaced by the fields of C-sets in uncountable Polish spaces. Some point of difficulty is to verify condition $(**)$. Fortunately, for the ideal of meager sets and for the ideal of sets having measure zero, the results of Vaught ([11]; for category), Kechris ([6]), Burgess ([2]) and Srivatsa ([10]) make it possible to overcome that trouble. Thus we establish the σ-saturation of the mixed measure-category products of ideals for the case of C-sets. We do not think that our result can be deduced directly from Gavalec's, since the field of Borel sets is properly included in the field of C-sets. However, we do not know whether the restrictions of the mixed measure-category product to Borel sets and to C-sets are distinct. That problem is discussed in the last section.

1. Some results of Gavalec. Let us recall some basic definitions from [5]. Our terminology is slightly modified.

A family $\mathcal{I}$ contained in a σ-field $\mathcal{S} \subset \mathcal{P}(X)$ is called a *σ-quasi-ideal* if

(i) for any $A, B \in \mathcal{S}$, $A \subset B$, $B \in \mathcal{I}$, we have $A \in \mathcal{I}$,
(ii) for any $A_1, A_2, \ldots \in \mathcal{I}$, $A_1 \subset A_2 \subset \ldots$, we have $\bigcup_{n=1}^\infty A_n \in \mathcal{I}$.

A σ-quasi-ideal $\mathcal{I}$ is called *proper* if

(iii) $\emptyset \neq \mathcal{I} \neq \mathcal{S}$.

A system $\mathcal{F}$ of σ-quasi-ideals in $\mathcal{S}$ is called *σ-quasi-maximal* if

(iv) for each $\mathcal{I} \in \mathcal{F}$ there is a countable $\mathcal{F}_0 \subset \mathcal{F}$ such that for each $A \in \mathcal{S}$ the conditions "$A \notin \mathcal{I}$" and "$X \setminus A \in \bigcup \mathcal{F}_0$" are equivalent.

The system $\mathcal{F}$ is called *closed* if

(v) for each finite $\mathcal{F}_0 \subset \mathcal{F}$ we have $\bigcup \mathcal{F}_0 \in \mathcal{F}$ and $\bigcap \mathcal{F}_0 \in \mathcal{F}$.

The theorems of this section are quoted from [5] with the σ-fields of Borel sets replaced by arbitrary σ-fields, since it is easily observed that the proofs from [5] can then be repeated.

(1.1) THEOREM (cf. [5], Th. 1.1). *If Q is a σ-quasi-maximal system of proper σ-quasi-ideals in a σ-field $\mathcal{S} \subset \mathcal{P}(X)$, then*

$$\overline{Q} = \{\bigcup \{\bigcap d : d \in \mathcal{D}\} : \mathcal{D} \in \mathcal{P}^{<\omega}(\mathcal{P}^{<\omega}(Q))\}$$

is the least closed σ-quasi-maximal system of proper σ-quasi-ideals in $\mathcal{S}$ extending Q.

A σ-quasi-ideal $\mathcal{I}$ in a σ-field $\mathcal{S}$ is called (i,j)-*quasi-saturated* (where i and j belong to the set $\mathbb{N}$ of all positive integers) if each family $\mathcal{F} \subset \mathcal{S}$ of cardinality $\geq j$ such that $X \setminus A \in \mathcal{I}$ for all $A \in \mathcal{F}$ contains a subfamily $\mathcal{F}_0$ of cardinality $\geq i$ such that $\bigcap \mathcal{F}_0$ is nonempty. We say that $\mathcal{I}$ is σ-*quasi-saturated* in $\mathcal{S}$ if for each $i \in \mathbb{N}$ there is $j \in \mathbb{N}$ such that $\mathcal{I}$ is (i,j)-quasi-saturated.

Let further $\mathcal{S}_1 \subset \mathcal{P}(X_1)$, $\mathcal{S}_2 \subset \mathcal{P}(X_2)$, $\mathcal{S} \subset \mathcal{P}(X_1 \times X_2)$ be fixed σ-fields. Assuming that $\mathcal{I}_1$, $\mathcal{I}_2$ are σ-quasi-ideals in $\mathcal{S}_1$, $\mathcal{S}_2$, respectively, satisfying $(*)$, $(**)$, we define $\mathcal{I}_1 \times \mathcal{I}_2$ by $(***)$.

(1.2) THEOREM (cf. [5], Th. 2.1). *If Q_1, Q_2 are σ-quasi-maximal systems of proper σ-quasi-ideals in $\mathcal{S}_1$, $\mathcal{S}_2$, respectively, such that conditions $(*)$, $(**)$ are satisfied for each $\mathcal{I}_2 \in Q_2$ and if, moreover, Q_1 is closed, then*

$$Q = \{\mathcal{I}_1 \times \mathcal{I}_2 : \mathcal{I}_1 \in Q_1, \mathcal{I}_2 \in Q_2\}$$

is a σ-quasi-maximal system of proper σ-quasi-ideals in $\mathcal{S}$.

(1.3) Remark. It is easy to verify that if, moreover, $\mathcal{I}_1 \in Q_1$, $\mathcal{I}_2 \in Q_2$ are ideals (thus σ-ideals), then $\mathcal{I}_1 \times \mathcal{I}_2$ is a σ-ideal.

(1.4) THEOREM (cf. [5], Th. 2.3). *If Q_1, Q_2 are σ-quasi-maximal systems of proper σ-quasi-saturated σ-quasi-ideals in $\mathcal{S}_1$, $\mathcal{S}_2$, respectively, such that conditions $(*)$, $(**)$ are satisfied for each $\mathcal{I}_2 \in Q_2$, then for any $\mathcal{I}_1 \in Q_1$, $\mathcal{I}_2 \in Q_2$ the product $\mathcal{I}_1 \times \mathcal{I}_2$ is σ-quasi-saturated in $\mathcal{S}$. If, moreover, $\mathcal{I}_1 \times \mathcal{I}_2$ is an ideal, then it is σ-saturated.*

2. The main result. Assume that X, Y are uncountable Polish spaces. Denote by $\mathcal{K}$ the family of all meager subsets of X. Let μ be a complete Borel measure on Y satisfying $0 < \mu(Y) < \infty$. Denote by $\mathcal{L}$ the family of all subsets of Y having measure zero. Recall that every C-set in a Polish space has the Baire property and is measurable with respect to any complete σ-finite Borel measure (cf. [8], pp. 110–111).

Now, we associate with the ideals $\mathcal{K}$ and $\mathcal{L}$ (analogously to [5]) certain σ-quasi-maximal systems of proper σ-quasi-ideals in $\mathcal{C}(X)$ and $\mathcal{C}(Y)$, respectively.

Let $\{U_n\}$ $(n = 0, 1, \ldots)$ be a fixed countable base of open sets in X. We may assume that $U_0 = X$. Let K consist of the families

$$\mathcal{K}^n = \{A \in \mathcal{C}(X) : A \cap U_n \text{ is meager in } U_n\}, \quad n = 0, 1, \ldots$$

Then K is a σ-quasi-maximal system of proper σ-quasi-ideals in $\mathcal{C}(X)$. It can be extended to a closed system $\overline{K}$, according to Theorem (1.1). Note that $\mathcal{K} \cap \mathcal{C}(X) = \mathcal{K}^0 \in \overline{K}$.

Assume that $\mu(Y) = a$ and let L consist of the families

$$\mathcal{L}^r = \{A \in \mathcal{C}(Y) : \mu(A) \le r\}$$

where r runs over all rationals from $[0, a)$. It is easy to verify that L is a closed σ-quasi-maximal system of proper σ-quasi-ideals in $\mathcal{C}(Y)$. In particular, we have $\mathcal{L} \cap \mathcal{C}(Y) = \mathcal{L}^0 \in L$.

To obtain our main result we need the following

(2.1) Proposition. *Assume that one of the following conditions holds*:

(a) $X_1 = X$, $X_2 = Y$, $\mathcal{S}_1 = \mathcal{C}(X)$, $\mathcal{S} = \mathcal{C}(X \times Y)$, $Q_2 = L$,
(b) $X_1 = Y$, $X_2 = X$, $\mathcal{S}_1 = \mathcal{C}(Y)$, $\mathcal{S} = \mathcal{C}(Y \times X)$, $Q_2 = \overline{K}$.

Then $(**)$ *holds for each* $\mathcal{I}_2 \in Q_2$.

This follows immediately from [10], Corollaries 2.7 and 3.10 (cf. also [2], [6], [11]).

Analogously to [5], Th. 2.4, one can prove

(2.2) Proposition. *Elements of the systems* $\overline{K}$, L *are* σ*-quasi-saturated.*

Finally, by combining Theorem (1.4) with Propositions (0.1), (2.1), (2.2) and Remark (1.3), we get

(2.3) Theorem. *Assume that one of the following conditions holds*:

(a*) $X_1 = X$, $X_2 = Y$, $\mathcal{S}_1 = \mathcal{C}(X)$, $\mathcal{S}_2 = \mathcal{C}(Y)$, $Q_1 = \overline{K}$, $Q_2 = L$, $\mathcal{S} = \mathcal{C}(X \times Y)$,
(b*) $X_1 = Y$, $X_2 = X$, $\mathcal{S}_1 = \mathcal{C}(Y)$, $\mathcal{S}_2 = \mathcal{C}(X)$, $Q_1 = L$, $Q_2 = \overline{K}$, $\mathcal{S} = \mathcal{C}(Y \times X)$.

Then for any $\mathcal{I}_1 \in Q_1$, $\mathcal{I}_2 \in Q_2$ *the product* $\mathcal{I}_1 \times \mathcal{I}_2$ *is* σ*-quasi-saturated in* $\mathcal{S}$. *If*, *moreover*, $\mathcal{I}_1$, $\mathcal{I}_2$ *are ideals*, *then* $\mathcal{I}_1 \times \mathcal{I}_2$ *is a* σ*-saturated* σ*-ideal in* $\mathcal{S}$.

From the last assertion of Theorem (2.3) it follows, in particular, that the products $\mathcal{K}^0 \times \mathcal{L}^0$, $\mathcal{L}^0 \times \mathcal{K}^0$ are σ-saturated in the respective fields of

C-sets. Furthermore, a standard argument allows us to extend the above to the case where the measure is σ-finite.

(2.4) THEOREM. *Assume that μ is a complete σ-finite Borel measure on Y and $\mathcal{K}$, $\mathcal{L}$ are defined as above. Let $\mathcal{K}^0 = \mathcal{K} \mid \mathcal{C}(X)$, $\mathcal{L}^0 = \mathcal{L} \mid \mathcal{C}(Y)$. Then $\mathcal{K}^0 \times \mathcal{L}^0$ and $\mathcal{L}^0 \times \mathcal{K}^0$ are σ-saturated σ-ideals in $\mathcal{C}(X \times Y)$ and $\mathcal{C}(Y \times X)$, respectively.*

(2.5) Remark. Similarly to [5] (where only the Borel case is considered) we may, by Theorem (1.2) and Proposition (2.1), construct any finite products of σ-quasi-ideals of C-sets associated either with measure or with category in Polish spaces. Moreover, it is easy to verify the associativity of producting. Thus, by applying Theorem (1.4) inductively, we can get a general version of Theorem (2.3) for any finite number of factors when each of them is associated either with measure or with category.

3. Some problems. According to the terminology of [8], the symbols Σ^1_1, Π^1_1 will stand for the point-classes of analytic and coanalytic sets, respectively. It is well known that for any uncountable Polish space Z all subsets of Z belonging to $\Sigma^1_1 \cup \Pi^1_1$ are C-sets (moreover, they belong to $\mathcal{C}_1(Z)$ in the hierarchy described in Section 0, and $\mathcal{C}_1(Z) \neq \mathcal{C}(Z)$; cf. [2]). Further, for any point-class $\mathcal{F}$ and an ideal $\mathcal{I} \subset \mathcal{P}(Z)$ we shall write briefly $\mathcal{I} \mid \mathcal{F}$ instead of $\mathcal{I} \mid (\mathcal{F} \cap \mathcal{P}(Z))$. We shall omit the letter Z in $\mathcal{B}(Z)$ and $\mathcal{C}(Z)$ since it will be clear which space Z is used.

Let X, Y, μ, $\mathcal{K}$, $\mathcal{L}$ have the meaning from Theorem (2.4). Define (cf. [4])

$$\mathcal{K}^* = \{E \subset Y \times X : E_y \in \mathcal{K} \text{ for all } y \in Y\},$$
$$\mathcal{L}^* = \{E \subset X \times Y : E_x \in \mathcal{L} \text{ for all } x \in X\}.$$

Observe that $\mathcal{K}^*$ and $\mathcal{L}^*$ are σ-ideals included in $\mathcal{L} \times \mathcal{K}$ and $\mathcal{K} \times \mathcal{L}$, respectively. From Section 2 of [4] it follows that

$$\mathcal{K}^* \mid \mathcal{B} = \mathcal{K}^* \mid \Sigma^1_1 \subsetneq \mathcal{K}^* \mid \Pi^1_1, \qquad \mathcal{L}^* \mid \mathcal{B} = \mathcal{L}^* \mid \Sigma^1_1 \subsetneq \mathcal{L}^* \mid \Pi^1_1.$$

We conjecture that the analogous properties for $\mathcal{K} \times \mathcal{L}$ and $\mathcal{L} \times \mathcal{K}$ hold. One part of this conjecture is settled by the following proposition based on ideals from [4].

(3.1) PROPOSITION.

(a) $(\mathcal{K} \times \mathcal{L}) \mid \mathcal{B} = (\mathcal{K} \times \mathcal{L}) \mid \Sigma^1_1$,
(b) $(\mathcal{L} \times \mathcal{K}) \mid \mathcal{B} = (\mathcal{L} \times \mathcal{K}) \mid \Sigma^1_1$.

Proof. We shall show (a); the proof of (b) is analogous. It is enough to verify the inclusion $(\mathcal{K} \times \mathcal{L}) \mid \Sigma^1_1 \subset (\mathcal{K} \times \mathcal{L}) \mid \mathcal{B}$. Consider any $A \in (\mathcal{K} \times \mathcal{L}) \mid \Sigma^1_1$. Then there are $B \in \mathcal{L}^*$ and $C \in \mathcal{K}$ such that $A \subset B \cup (C \times Y) \in \Sigma^1_1$. We may assume that C is Borel and B, $C \times Y$ are disjoint. Thus $B \in \Sigma^1_1$. By

[3], Th. 4.2, there is $B^* \in \mathcal{B} \cap \mathcal{L}^*$ including B. Hence $A \subset B^* \cup (C \times Y) \in (\mathcal{K} \times \mathcal{L}) \,|\, \mathcal{B}$, which ends the proof.

(3.2) PROBLEM. Are the properties

$$(\mathcal{K} \times \mathcal{L}) \,|\, \Pi^1_1 \neq (\mathcal{K} \times \mathcal{L}) \,|\, \Sigma^1_1 , \qquad (\mathcal{L} \times \mathcal{K}) \,|\, \Pi^1_1 \neq (\mathcal{L} \times \mathcal{K}) \,|\, \Sigma^1_1$$

true? If so, then, by Proposition (3.1), we would have

$$(\mathcal{K} \times \mathcal{L}) \,|\, \mathcal{B} \subsetneq (\mathcal{K} \times \mathcal{L}) \,|\, \mathcal{C} , \qquad (\mathcal{L} \times \mathcal{K}) \,|\, \mathcal{B} \subsetneq (\mathcal{L} \times \mathcal{K}) \,|\, \mathcal{C} .$$

(Note that $(\mathcal{K} \times \mathcal{L}) \,|\, \mathcal{B} \subsetneq \mathcal{K} \times \mathcal{L}$ and $(\mathcal{L} \times \mathcal{K}) \,|\, \mathcal{B} \subsetneq \mathcal{L} \times \mathcal{K}$ by [7], Th. 1.3.)

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