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ON HERBRAND'S THEOREM

BY

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The Herbrand theorem [1] asserts that with every formula α_0 (of the classical predicate calculus) in the normal prenex form we can associate an infinite sequence $\alpha_1, \alpha_2, \ldots$ of formulas without quantifiers such that α_0 is provable if and only if one of the formulas α_n $(n = 1, 2, \ldots)$ is provable.

In this note I shall deal only with a part of the Herbrand theorem, viz. with the following implication:

(H_1) *If α_0 is provable, then one of the formulas α_n $(n = 1, 2, \ldots)$ is provable.*

The construction of formulas α_n is rather special. I recall it by the following example. *E. g.*, let α_0 be the formula

$$\sum_x \prod_y \sum_z \prod_u \beta_0(x, y, z, u) \tag{1}$$

where the formula β_0 contains no quantifier. Let $V = (x_1, x_2, \ldots)$ be the set of all individual variables(1). Let f_0 be a fixed mapping of V into V, and let g_0 be a fixed mapping of $V \times V$ into V. Then the Herbrand formula α_n $(n = 1, 2, \ldots)$ is the following one:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \beta_0\big(x_i, f_0(x_i), x_j, g_0(x_i, x_j)\big), \tag{2}$$

and the considered part (H_1) of the Herbrand theorem asserts that if (1) is provable, then one of the formulas (2) is provable (the remaining part (H_2) of the Herbrand theorem asserts that, under some additional hypothesis about f_0 and g_0, if one of the formulas (2) is provable, so is (1)).

(1) For simplicity we assume that the predicate calculus under consideration contains no terms except individual variables. In the opposite case some obvious modifications are necessary.

We see that the Herbrand formulas α_n have the form of disjunction:

$$\alpha_n = \beta_1 + \ldots + \beta_n, \tag{3}$$

viz. they are a partial disjunction of an "infinite disjunction"

$$\beta_1 + \beta_2 + \ldots \tag{4}$$

Roughly speaking, if α_0 is provable, then (4) is also "provable", and, by (H_1), there exists a positive integer n such that (3) is provable.

In the usual set-theoretical interpretation of the classical predicate calculus, the logical formulas are interpreted as some subsets of a fixed space X, the provable formulas are interpreted as the whole space X, and the logical disjunction is interpreted as the set-theoretical union. Therefore the part (H_1) of the Herbrand theorem reminds us of the following well-known theorem of Borel on compact topological spaces:

(B) *If X is compact and $X = B_1 + B_2 + \ldots$, where B_n are open, then there exists an integer n such that $X = B_1 + \ldots + B_n$.*

The purpose of this paper is to show that (H_1) is a direct consequence of (B). More exactly, (H_1) is a simple consequence of the Borel theorem and of a representation theorem of Rieger (see [4], [5] and also [3]). I recall here only the following immediate consequence of Rieger's theorem:

(R) *With every formula α of the predicate calculus under consideration we can associate a (Borel) subset (denoted here by $\|\alpha\|$) of the Cantor discontinuum C in such a way that:*

(i) $\|\alpha_1 + \ldots + \alpha_n\| = \|\alpha_1\| + \ldots + \|\alpha_n\|$;

(ii) $\left\| \sum_{a} \alpha(x) \right\| = \sum_{i=1}^{\infty} \|\alpha(x_i)\|, \quad \left\| \prod_{a} \alpha(x) \right\| = \prod_{i=1}^{\infty} \|\alpha(x_i)\|$;

(iii) *if α contains no quantifier, then the set $\|\alpha\|$ is open in C;*

(iv) *α is provable if and only if $\|\alpha\| = C$.*

In fact, suppose that the formula α_0 (see (1)) is provable. By (iv), $\|\alpha_0\| = C$, *i. e.*

$$\left\| \sum_{x} \prod_{y} \sum_{z} \prod_{u} \beta_0(x, y, z, u) \right\| = C.$$

It follows from (ii) that

$$\sum_{i=1}^{\infty} \prod_{k=1}^{\infty} \sum_{j=1}^{\infty} \prod_{l=1}^{\infty} \|\beta_0(x_i, x_k, x_j, x_l)\| = C.$$

Consequently, by the well-known rules of distributivity for sets,

$$\prod_{f \in \Phi_1} \prod_{g \in \Phi_2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \|\beta_0(x_i, f(x_i), x_j, g(x_i, x_j))\| = C$$

where Φ_1 is the class of all mappings of V into V and Φ_2 is the class of all mappings of $V \times V$ into V. In other words, we have

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \|\beta_0(x_i, f(x_i), x_j, g(x_i, x_j))\| = C$$

for every $f \in \Phi_1$ and $g \in \Phi_2$. Consequently

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \|\beta_0(x_i, f_0(x_i), x_j, g_0(x_i, x_j))\| = C.$$

Since β_0 contains no quantifier, it follows from (iii) and the Borel theorem that there exists an integer n such that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \|\beta_0(x_i, f_0(x_i), x_j, g_0(x_i, x_j))\| = C.$$

By (i), we have

$$\left\| \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_0(x_i, f_0(x_0), x_j, g_0(x_i, x_j)) \right\| = C,$$

i. e. $\|\alpha_n\| = C$. This proves, however (see (iv)), that α_n is provable.

Notice that this set-theoretical proof is valid also for open theories (in the Rieger theorem, C denotes then a compact totally disconnected space, which does not alter the proof).

The proof of the second part (H_2) of the Herbrand theorem can also be translated into the set-theoretical language. However, it seems from the known proofs (see *e. g.* [2]) of (H_2) that the substitution rule plays an essential part in proving (H_2). Since the logical substitution rule has no analogue in the General Theory of Sets, this translation does not seem to be convenient.

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ON A COMBINATORICAL PROBLEM

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1. This paper deals with the following problem:

A matrix is said *to be of type* $(R, K)_s$, where $0 \leqslant s \leqslant RK$, if it has R rows and K columns and if s of its elements are 1 and the rest 0. Now let R, K, r, k, where $1 \leqslant r \leqslant R$ and $1 \leqslant k \leqslant K$, be four given natural numbers. Which is the greatest number $s = A(R, K, r, k)$, such that there exists a matrix of the type $(R, K)_s$, which does not contain any minor of the type $(r, k)_{rk}$, *i. e.* a minor with r rows and k columns and all elements equal to 1?

This problem was (for $R = K$, $r = k$) raised by K. Zarankiewicz in [3]. It is properly a logical problem and can be formulated in the following way: Let E and F be two sets with R and K elements, respectively. How many elements s can a relation between E and F (*i. e.* a subset G of $E \times F$) contain, without containing any subset of the type $E' \times F'$, where E' and F' are subsets of E and F with r and k elements, respectively? In the following, however, we use the matrix formulation.

In [2] T. Kővari, V. T. Sós and P. Turán proved that

$$A(n, n, j, j) < jn + [(j-1)^{1/j} n^{(2j-1)/j}], \tag{1}$$

where $[x]$ denotes the integral part of x. They also showed the asymptotic formula

$$\lim_{n\to\infty} A(n, n, 2, 2) n^{-3/2} = 1. \tag{2}$$

The same method as was used in [2] to prove (1) can also, as mentioned there, be used to give an estimate of $A(R, K, r, k)$. This gives

$$A(R, K, r, k) \leqslant (r-1)K + (k-1)^{1/r} K^{1-1/r} R \tag{3}$$

after a slight sharpening of the estimates influencing the first term in the second member.

In this paper I will in section 2 give a special method for estimating $A(R, K, 2, k)$ from above, which gives another estimate than that