COLLOQUIUM MATHEMATICUM

VOL. VI DÉDIÉ À M. CASIMIR KURATOWSKI 1958

ON ORBITS OF HOMEOMORPHISMS

BY

S. KINOSHITA (OSAKA)

Let X be a metric space and f a continuous mapping of X into itself. Consider the positive half-orbit

$$O_+(x, f) = \{f(x), f^2(x), \ldots, f^n(x), \ldots\}$$

for each $x \in X$ and put

$$P_+(f) = \{x | \overline{O_+(x, f)} = X\}, \quad Q_+(f) = X - P_+(f).$$

If X is a circumference of a circle, then there exists a homeomorphism h of X onto itself such that $P_+(h) = X$. If h is a homeomorphism of the Euclidean plane E^2 onto itself, then the sets $P_+(h)$ and $Q_+(h)$ can be more complicated. For instance A. S. Besicovitch [1] gave a homeomorphism h of E^2 onto itself such that $P_+(h) \neq 0$ and, of course, $Q_+(h) \neq 0$. His type of homeomorphisms has also been studied in [2], [3]. On the other hand the author proved with T. Homma [5] that if X is a locally compact, non compact metric space, then $Q_+(f)$ is dense in X for every continuous mapping f (see also [3], p. 202).

In this paper assuming X to be compact, we shall deal with the sets $P_+(f)$ and $Q_+(f)$ more systematically. In fact we shall prove the following

THEOREM 1. *Let X be a compact metric space and f a continuous mapping of X into itself. If $P_+(f) \neq 0$, then $P_+(f)$ is a dense G_δ set.*

THEOREM 2. *Let X be a compact metric space and f a continuous mapping of X into itself. If $Q_+(f) \neq 0$, then $Q_+(f)$ is a dense F_σ set.*

An immediate consequence of Theorem 2 is the following

COROLLARY 1. *Let X be a compact metric space and f a continuous mapping of X into itself. If f has at least one fixed point, then $Q_+(f)$ is a dense F_σ set.*

Further

COROLLARY 2. *If h be a homeomorphism of an n-dimensional Euclidean space E^n onto itself, then $Q_+(h)$ is a dense F_σ set* (see [5], p. 371).

Furthermore if h is a homeomorphism of X onto itself, then we can define the negative half-orbit $O_-(x,h)$ and the whole orbit $O(x,h)$ as follows:

$$O_-(x,h) = O_+(x,h^{-1}) \quad \text{and} \quad O(x,h) = \sum_{m=0}^{\pm\infty} h^m(x).$$

Now put $P_-(h) = P_+(h^{-1})$ and $Q_-(h) = Q_+(h^{-1})$. Furthermore put $P(h) = P_+(h)\cdot P_-(h)$ and $Q(h) = \{x \mid \overline{O(x,h)} \neq X\}$. Then we shall prove the following

THEOREM 3. *Let X be a compact metric space and h a homeomorphism of X onto itself. If $P_+(h) \neq 0$, then $P(h)$ is a dense G_δ set.*

THEOREM 4. *Let X be a compact metric space and h a homeomorphism of X onto itself. If $Q_+(h) \neq 0$, then $Q(h) \neq 0$.*

An immediate consequence of Theorem 4 is

COROLLARY 3. *Let X be a compact metric space and h a homeomorphism of X onto itself. If $Q_+(h) \neq 0$, then $Q_+(h)\cdot Q_-(h) \neq 0$.*

1. Hereafter we shall always assume that X is a compact metric space. Let f be a continuous mapping of X into itself. A subset Y of X is said to be *ε-dense in X*, if for each $x \in X$ there exists a $y \in Y$ such that $d(x,y) < \varepsilon$ [1]. In other words Y is ε-dense in X if and only if $\{U(y,\varepsilon)\}_{y\in Y}$ [2] is an open covering of X. Then a subset $P_+(\varepsilon, f)$ of X is defined as follows: A point $x \in X$ is contained in $P_+(\varepsilon, f)$ if and only if $O_+(x,f)$ is ε-dense in X.

LEMMA 1. *If $x \in P_+(\varepsilon, f)$, then there exists a natural number N such that $\sum_{n=1}^{N} f^n(x)$ is ε-dense in X.*

Proof. This follows immediately from the compactness of X.

LEMMA 2. *$P_+(\varepsilon, f)$ is an open subset of X.*

Proof. If $P_+(\varepsilon, f)$ is empty, then our lemma is obvious. Now let $x \in P_+(\varepsilon, f)$ and suppose by Lemma 1 that $\sum_{n=1}^{N} f^n(x)$ is ε-dense in X. Put $Y = \sum_{n=1}^{N} f^n(x)$. Then $d(x, Y)$ is a continuous real-valued function on X. Since X is compact,

$$c = \max_{x\in X} d(x, Y)$$

exists and $0 \leqslant c < \varepsilon$. Put $\varepsilon - c = d$. If δ is a sufficiently small positive number, then for each $y \in U(x,\delta)$ all $d(f(x), f(y)), d(f^2(x), f^2(y)), \ldots$

(1) $d(x,y)$ means the distance from x to y.

(2) $U(y,\varepsilon) = \{x \mid d(x,y) < \varepsilon\}$.

$d(f^N(x), f^N(y))$ are smaller than d. Now we are only to prove that $y \in P_+(\varepsilon, f)$. If $z \in X$, then there exists an n_0 $(1 \leqslant n_0 \leqslant N)$ such that, $d(z, f^{n_0}(x)) \leqslant c$. Therefore

$$d(z, f^{n_0}(y)) \leqslant d(z, f^{n_0}(x)) + d(f^{n_0}(x), f^{n_0}(y)) < c + d = \varepsilon.$$

Thus our proof is complete.

LEMMA 3. *$P_+(f)$ is a G_δ set.*

Proof. It is easy to see that

$$P_+(f) = \prod_{n=1}^{\infty} P_+\left(\frac{1}{n}, f\right).$$

Then our statement is obvious by Lemma 2.

LEMMA 4. *If $x \in P_+(f)$, then $f(x) \in P_+(f)$.*

Proof. Since

$$X = \overline{O_+(x,f)} = \overline{f(x) + O_+(f(x),f)} = f(x) + \overline{O_+(f(x),f)},$$

we are only to prove that $f(x) \in \overline{O_+(f(x),f)}$. From $x \in \overline{O_+(x,f)}$ it follows that

$$f(x) \in f\overline{(O_+(x,f))} \subset \overline{f(O_+(x,f))} = \overline{O_+(f(x),f)}.$$

Thus the proof is complete.

Proof of Theorem 1. From Lemma 4 it follows that if $x \in P_+(f)$, then $O_+(x,f) \subset P_+(f)$. Since $X = \overline{O_+(x,f)} \subset \overline{P_+(f)}$, our statement is obvious by Lemma 3.

2. Proof of Theorem 2. It is trivial by definition that $Q_+(f)$ is an F_σ set. Now let A be an open non-empty subset of X such that $\bar{A}$ is compact. Since X is compact, we are only to prove that $\bar{A}Q_+(f) \neq 0$. Consider the sequence $\bar{A}, f(\bar{A}), \ldots, f^n(\bar{A}), \ldots$

First suppose that there exists a natural number N such that

$$f^{N+1}(\bar{A}) \subset \sum_{n=0}^{N} f^n(\bar{A}).$$

Since $\sum_{n=0}^{N} f^n(\bar{A})$ is compact, if $\sum_{n=0}^{N} f^n(\bar{A}) \neq X$, then $\bar{A} \subset Q_+(f)$. Therefore our theorem is obvious. If $\sum_{n=0}^{N} f^n(\bar{A}) = X$, then there exists a point $x \in \bar{A}$ such that $f^{n_0}(x) \in Q_+(f)$ for some n_0 $(0 \leqslant n_0 \leqslant N)$. Then $x \in Q_+(f)$ by Lemma 4 and the proof of the first case is complete.

Now suppose that for each n

$$f^{n+1}(\bar{A}) \not\subset \sum_{i=0}^{n} f^i(\bar{A}).$$

Then the sequence $\{f^n(\bar{A})\}$ is said to be a *bulging sequence* and it is proved that there exists a point $x \in \bar{A}$ such that $f^n(x) \bar{\varepsilon} A$ for every n $(n \geqslant 1)$ (see [5], § 1). Then $x \in Q_+(f)$, and the proof of our theorem is complete.

3. Now let h be a homeomorphism of X onto itself. Put

$$P_-(\varepsilon, h) = P_+(\varepsilon, h^{-1}).$$

LEMMA 5. *If $P_+(h) \neq 0$, then $P_-(h)$ is a dense G_δ set.*

Proof. First we shall prove that $P_-(\varepsilon, h)$ is dense in X for every $\varepsilon > 0$. Let $x \in P_+(h)$. Then by Lemma 1 there exists a natural number N such that $\sum_{n=1}^{N} h^n(x)$ is ε-dense in X. Then all $h^{N+n}(x)$ are contained in $P_-(\varepsilon, h)$ for every n $(\geqslant 1)$. On the other hand, since $x \in P_+(h)$, $h^N(x) \in P_+(h)$ by Lemma 4. Then $\overline{\sum_{n=1}^{\infty} h^{N+n}(x)}$ is dense in X. Therefore $P_-(\varepsilon, h)$ is dense in X.

Then from Lemma 2 it follows that $P_-(\varepsilon, h)$ is a dense open subset of X. Since

$$P_-(h) = \prod_{n=1}^{\infty} P_-\left(\frac{1}{n}, h\right),$$

$P_-(h)$ is a dense G_δ set, and the proof is complete.

Proof of Theorem 3. Since $P_+(h)$ and $P_-(h)$ are dense G_δ sets and $P(h) = P_+(h) \cdot P_-(h)$, our statement is obvious.

4. Proof of Theorem 4. Let $x \in Q_+(h)$. By definition

$$\overline{\sum_{n=1}^{\infty} h^n(x)} \neq X.$$

Consider the sequence $\{h^n(x)\}$. Since X is compact, there exists a point y such that y is one of the limit points of $\{h^n(x)\}$. Then there exists a subsequence $\{h^{n_i}(x)\}$ of $\{h^n(x)\}$ which converges to y.

Now we prove that $y \in Q(h)$. Let m be an integer. Then the sequence $\{h^{m+n_i}(x)\}$ converges to $h^m(y)$ for every m. Therefore

$$h^m(y) \in \overline{\sum_{n=1}^{\infty} h^n(x)} \quad \text{for every } m.$$

Thus we have proved that $\overline{\sum_{m=0}^{\pm\infty} h^m(y)} \subset \overline{\sum_{n=1}^{\infty} h^n(x)}$. Since $\overline{\sum_{n=1}^{\infty} h^n(x)} \neq X$, our proof is complete.

REFERENCES

[1] A. S. Besicovitch, *A problem on topological transformations of the plane*, Fundamenta Mathematicae 28 (1937), p. 61-65.

[2] — *A problem on topological transformations of the plane II*, Proceedings of the Cambridge Philosophical Society 47 (1951), p. 38-45.

[3] G. D. Birkhoff, *Dynamical systems*, American Mathematical Society Colloquium Publications 9, New York 1927.

[4] G. D. Hedlund, *A class of transformations of the plane*, Proceedings of the Cambridge Philosophical Society 51 (1955), p. 554-564.

[5] T. Homma and S. Kinoshita, *On the regularity of homeomorphisms of E^n*, Journal of the Mathematical Society of Japan 5 (1953), p. 365-371.

DEPARTMENT OF MATHEMATICS, OSAKA UNIVERSITY

Reçu par la Rédaction le 15. 10. 1958