pend pas du choix des polygonales $P_\varepsilon, P'_\varepsilon$: si on leur substitue deux autres lignes polygonales orientées $p_\varepsilon, p'_\varepsilon$ et si l'on pose $q_\varepsilon = p_\varepsilon + p'_\varepsilon$, on aura

$$(Q_\varepsilon, q_\varepsilon) = \|Q+(-1)q_\varepsilon\|$$
$$= \|(C+(-1)p_\varepsilon)+(C'+(-1)p'_\varepsilon)+(-1)(C+(-1)P_\varepsilon)+$$
$$+(-1)(C'+(-1)P'_\varepsilon)\| \leqslant 4\varepsilon.$$

D'où $(\sigma, q_\varepsilon) \leqslant (\sigma, Q_\varepsilon)+4\varepsilon$, donc $q_\xi \to \sigma$, comme Q_ε, quand $\varepsilon \to 0$.

Dès lors, si l'on a pu définir la somme de deux lignes polygonales orientées, alors, si, pour deux courbes de Jordan C, C', on détermine des lignes polygonales approchées $P_\varepsilon, P'_\varepsilon$, telles que 1° leur somme Q_ε tend vers une courbe limite unique σ qui ne dépend pas du choix de $P_\varepsilon, P'_\varepsilon$, 2° la somme de C et de C' ne peut être que cette courbe limite σ.

On est ainsi ramené à résoudre notre problème *pour le cas théoriquement plus simple* où l'espace Γ des courbes de Jordan est remplacé par son sous-espace γ, constitué par *les lignes polygonales orientées* (en rappelant toutefois que γ n'est pas complet).

TRAVAUX CITÉS

[1] S. Banach, *Un théorème sur les transformations biunivoques*, Fundamenta Mathematicae 6 (1924), p. 236-239.

[2] M. Fréchet, *Les espaces abstraits et leur théorie considérés comme introduction à l'analyse générale*, Paris 1928.

[3] — *Les mathématiques et le concret*, Paris 1952.

[4] — *Sur quelques points du Calcul fonctionnel (Thèse)*, Paris 1906.

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IS w A FUNCTION OF u?

BY

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1. Quantities, Fluents, Functions. By *quantity* I mean an ordered pair whose second member (or *value*) is a number while its first member (or *object*) may be anything. (Here and in the sequel, "number" means real number). Two quantities Q and Q′ will be said to be *consistent* if and only if either *obj* Q ≠ *obj* Q′ or *obj* Q = *obj* Q′ and *val* Q = *val* Q′. (In the latter case, Q and Q′ are regarded as *equal*).

By a *fluent* I mean (reviving, in precise form, a half-forgotten term introduced by Newton) a class of pairwise consistent quantities(1). The class of all objects (all values) of a fluent will be referred to as its *domain* (its *range*). If A is any class, a fluent with the domain A is a class of quantities which, for each element α of A, includes exactly one quantity with the object α. The value of that quantity, if the fluent is called u, will be denoted by $u\alpha$. All fluents will be designated by letters in italic type.

As an example of a fluent, I mention the time in seconds elapsed since a certain instant, condidered as the class t of all quantities $(\tau, t\tau)$ for any act τ of reading a timer calibrated in seconds which showed 0 when it was started at the instant mentioned. Here, $t\tau$ denotes the number read as the result of the act τ.

A fluent whose domain is a class of numbers will be called a 1-place function — briefly, a *function* (since several-place-functions will not herein be considered)(2). Examples of functions include

log, the class of all pairs (x, *log* x) for any x > 0;

j^3, the class of all pairs $(\mathrm{t}, j^3\,\mathrm{t}) = (\mathrm{t}, \mathrm{t}^3)$ for any t.

Here, non-italicized letters x and t are used as number variables (3). Each replacement of those letters with a numeral yields the designation of specific elements of the functions defined, *e. g.*,

(2, *log* 2) and (2, 2^3), respectively.

The letter j and the name *identity* function will be permanently reserved for the class (4) of all pairs (x, x) for any number x.

2. Incomplete, unanswerable Questions. Let u and w be two fluents. Is $w = \sin u$? Is w some function of u? These questions are unanswerable because they are incomplete. Even with regard to specific fluents, say, the time t (previously defined) and p (the force in dynes acting on a specific particle), without further assumptions it is impossible either to assert or to deny that $p = sin t$.

If in the last formula, the letters p and t were interpreted as number variables — in the literature this misinterpretation is rather common — then the formula $p = sin t$ could not possibly be valid, since the statement that *any* value of the pressure is the sine of *any* value of the time is obviously false. The problem is not furthered by calling p a *dependent* variable — words that lack a clear definition but whose intent is the stipulation that p must *not* be replaced by just *any* value of the force. Since a number variable *may* be replaced by the designations of *any* number belonging to a certain class, the words *dependent variable*, however timehonored, *are a plain contradiction in terms* and, besides, lack any indication as to *which* values of p are the sine of *which* values of t.

In terms of their two *ranges*, the question as to whether one fluent is the sine (or a function at all) of another fluent clearly is unanswerable. It is necessary to make assumptions that somehow connect their two *domains*. True, in physics it has become customary to write $p = sin t$ if and only if the value read on a forcemeter is the sine of the value read on the timer for any two acts reading performed *simultaneously*. In mathematics, it has become traditional to call tangent and cotangent reciprocal because of the reciprocity of the values that these functions assume for any one and the *same* number.

But in discussing the general question, that is, the question concerning any two fluents whose domains are abstract classes, these principles of comparison are out of the question. Which elements of two abstract classes can be said to be *simultaneous*? How can two fluents whose domains are disjoint assume values for the *same* element? No pairing of two abstract classes is, or can ever become *traditional*. Nor is there any "*natural*" pairing.

3. Completion of the Questions by Relativization. Besides two fluents u and w, let there be given a subclass Π of the Cartesian product of their domains:

(1) $$\Pi \subseteq \mathrm{Dom}\,u \times \mathrm{Dom}\,w.$$

I will write

(2) $$w = sin\,u \ (\text{rel. } \Pi)$$

if and only if

$$w\beta = sin\,u\alpha \text{ for any } \alpha \text{ and } \beta \text{ such that } (\alpha, \beta)\,\varepsilon\,\Pi$$

or, which is equivalent, if and only if

$$(\alpha, \beta)\,\varepsilon\,\Pi \text{ implies that } (u\alpha, w\beta)\,\varepsilon\, sin,$$

where sin is the class of all pairs (x, sinx) for any number x.

More generally, for any two fluents u and w, any class Π satisfying (1), and any function f, I will write

(3) $$w = fu \ (\text{rel. } \Pi)$$

if and only if

$$w\beta = fu\alpha \text{ for any } \alpha \text{ and } \beta \text{ such that } (\alpha, \beta)\,\varepsilon\,\Pi.$$

The fluent w is said to be *a function* of the fluent u relative to Π if and only if there is a function f such that (3) holds. It is now possible to answer, in properly relativized form, the question in the title of this paper. Using the fact that any subclass of a function is a function([5]), one obtains

ANSWER I. *The fluent w is a function of the fluent u relative to the subclass Π of* Dom$u$×Dom$w$ *if and only if*

the class of all ordered pairs of numbers $(u\alpha, w\beta)$ for any α and β such that (α, β) belongs to Π is a function.

Using the fact that a class of ordered pairs of numbers is a function if and only if it does not include two inconsistent pairs (with equal first, and unequal second, members), one obtains

ANSWER II. *w is a function of u relative to Π if and only if $(\alpha, \beta)\varepsilon\Pi$, $(\alpha', \beta')\varepsilon\Pi$, and $u\alpha = u\alpha'$ imply $w\beta = w\beta'$.*

These answers make precise the century-old definition of one of the most important concepts of applied mathematics: *A variable quantity is a function of another variable quantity if, whenever the other assumes equal values, then so does the first.* This statement, as found in the literature lacks precision in two ways: (1) In some places, that traditional statement is not preceded by any definition of the words *variable quantity* (that is to say, their meaning is not explained by either an explicit definition or a set of postulates implicitly defining them), while, in other places, it is preceded by a definition of variable quantity as number variable (even though, as has been shown in Section 2, these two definitions contradict one another); (2) The traditional statement lacks an indispensable *relativization* which, in the preceding answers, is reflected in the reference to the class Π.

4. Examples. The following simple examples of functional connections between fluents are taken from the realm of functions. In such cases, Π is a class of pairs of numbers — traditionally, the class of all pairs of equal numbers, that is, the identity function j. It is relative to $\Pi = j$ that the 6th power is the 3rd power of the 2nd power; that is to say, if substitution is expressed by juxtaposition — the product of j^3 and j^2 would be denoted by $j^3 \cdot j^2$ — then

$$j^6 = j^3 j^2 \text{ means } j^6 = j^3 j^2 \text{ (rel. } j).$$

Functions considered as fluents may, however, be connected relative to classes other than j such as $2j$, that is, the class of all pairs $(x, 2x)$ for any x. For instance,

$$j^6 = 64 j^3 j^2 \text{ (rel. } 2j);$$

this means

$$\text{if } (x, y) \varepsilon 2j, \text{ then } j^6 y = 64 j^3 (j^2 x) \text{ or } y^6 = 64 j^3 x^2.$$

The class Π need not define a 1-1 mapping of numbers(6); it may be a 2-1 mapping as in

$$j^6 = j^{12} j^2 \text{ (rel. } j^4).$$

Nor is it necessary that the class Π be a function; Π may define a 1-2 mapping; for instance, Π may be the class of all pairs (x^2, x) for any x, which in the sequel will be denoted by (j^2, j). This class is the set-theoretical sum of two classes that are functions:

$j^{1/2}$, the class of all pairs $(x, \sqrt{x})$ for any $x \geqslant 0$;

$-j^{1/2}$, the class of all pairs $(x, -\sqrt{x})$ for any $x \geqslant 0$.

The sum class is used in the statement

$$j^3 = j^{3/2} j^4 \left(\text{rel. } (j^2, j)\right), \text{ that is, (rel. } j^{1/2} \cup -j^{1/2}).$$

While j^3 is a function of j^4 relative to $j^{1/2} \cup -j^{1/2}$, it is not a function of j^4 relative to j; in other words, j^3 is not in the traditional sense a function of j^4. Indeed, for j^3 to be a function of j^4 it would be necessary that the class of all pairs of numbers

$$(j^4 x, j^3 x) \quad \text{for any} \quad x$$

be a function. But this class is not a function since it includes, for instance, the inconsistent pairs

$$(j^4 1, j^3 1) = (1,1) \quad \text{and} \quad \left(j^4(-1), j^3(-1)\right) = (1,-1).$$

5. General Remarks. A constant fluent w (that is, a fluent whose range consists of a single number) is a function of any fluent relative to any subclass Π of $\mathrm{Dom}\, u \times \mathrm{Dom}\, w$. A non-constant fluent is not a function of any constant function relative to any Π.

If t_1 and t_2 are two functions with equal domains, then (t_1, t_2) may denote the class of all ordered pairs of numbers $(t_1 x, t_2 x)$ for any x belonging to that common domain. Without entering into details about the domains and ranges of the functions f, g, and h, one may say that

$$h = fg \text{ (rel. } (t_1, t_2)) \text{ if and only if } ht_2 = fgt_1 \text{ (rel. } j).$$

In particular, if t replaces t_2,

$$ht = fg \text{ (rel. } (j, t)), \text{ that is, (rel. } t) \text{ if and only if } ht = fg \text{ (rel. } j).$$

Especially,

$$ht = fj \text{ (rel. } t) \text{ if and only if } ht = f \text{ (rel. } j).$$

Thus, any function h is a function of j relative to any function t, namely, ht of j.

6. Piecewise Functional Dependence. If, in $\mathrm{Dom}\, u$, open subsets are defined, then w may be said to be a function of u relative to Π *in an open set* $\mathrm{U} \leqslant \mathrm{Dom}\, u$ if and only if the following condition is satisfied:

If α and $\alpha' \varepsilon \mathrm{U}$, (α, β) and $(\alpha', \beta') \varepsilon \Pi$, and $u\alpha = u\alpha'$, then $w\beta = w\beta'$.

For instance, if P (and N) denote the class of all positive (of all negative) numbers, then j^3 is a function of j^4 on P as well as on N. (Here and in the sequel, *function* means function in the ordinary sense, that is, relative to j). The function j is a function of the sine function in any interval not including an odd multiple of $\pi/2$.

For any element α_0 of $\mathrm{Dom}\, u$, the fluent w may by said to be a function of u relative to Π *near* α_0 if and only if α_0 is contained in an open subset U of $\mathrm{Dom}\, u$ such that w is a function of u relative to Π in U. For instance j^3 is a function of j^4 near any number $\neq 0$. But j^3 is not a function of j^4 near 0.

One might be tempted to define: w is a function of u *at* α if there is a neighborhood U of α such that

$$\alpha' \varepsilon \mathrm{U}, \ (\alpha, \beta) \text{ and } (\alpha', \beta') \varepsilon \Pi, \text{ and } u\alpha' = u\alpha \text{ imply } w\beta' = w\beta.$$

In this sense, j^3 would be a function of j^4 even at 0 (even though j^3 is not a function of j^4 near 0). But this definition would be misleading inasmuch as it does not imply the existence of a function connecting w with u anywhere.

7. Functionally Completely Unrelated Fluents. Piecewise dependence gives rise to the definition of a new and promising type of what might be called complete functional unrelatedness. Let u be a fluent in whose domain open subsets are defined. Let w be a fluent and Π a subset of $\mathrm{Dom}\,u\times\mathrm{Dom}\,w$ then w may be said to be *nowhere a function* of u relative to Π if and only if w is not a function of u relative to Π in any open subset of Dom u. In other words, w is nowhere a function of u if each open subset of Dom u contains two elements α' and α'' such that $u\alpha'=u\alpha''$ and that there exist two elements β' and β'' of Dom w for which (α',β') and (α'',β'') belong to Π and $w\beta'\neq w\beta''$.

For instance, j is nowhere a function (relative to j) of any constant function; and j is nowhere a function of Cantor's non-decreasing non-constant function h which is constant on all intervals that are complementary to Cantor's discontinuum. On the other hand, h is (just as is any function) a function of j relative to j. Hence the relation *being nowhere a function of* is not symmetric.

REFERENCES

(1) Cf. K. Menger, *Calculus. A Modern Approach*, (Chapter VII), Boston 1955

(2) The objects falling under the above-defined concept of fluent are those that Newton called fluents. The term function was introduced by Leibniz for connections between fluents — differentiable and integrable connections such as the trigonometric and logarithmic functions. Leibniz did not apply it to time and force, which cannot be either differentiated or integrated nor in any way describe connections between fluents since no fluents can be substituted into them. Those who propose to call all fluents "functions" are bound to experience the need for a word describing the particular functions of the type of the sine and the logarithm that lend themselves to substitution, that is, whose domains consist of numbers or, at any rate, of ordered classes of numbers.

(3) The reader will notice the advantages of the new typography introduced l. c. (1).

Type	in designations of	in ad hoc abbreviations	serving as
roman	numbers: 2, e, ...	say, e for $\sqrt{2}+\sqrt[3]{2}+\sqrt[4]{2}$	number variable: x, t, e, ...
italic	fluents: t, p, ... functions: 2, log, j^3, ...	say, u for $16\,t^2$ say, f for $2+log+j^3$	fluent variable: u, w, ... function variable: f, g, ...
bold face	operators: $\boldsymbol{D}$, $\boldsymbol{L}$, ...		operator variable: $\boldsymbol{T}$, ...

An operator is a class of consistent pairs of functions. For instance, the derivative $\boldsymbol{D}$ is the class of pairs $(f,\boldsymbol{D}f)$ for any differentiable function. The class $\boldsymbol{D}$

includes, *e. g.*, the pairs (log, j^{-1}) and (sin, cos). The Laplace transform $\boldsymbol{L}$ is the class of all pairs $(f,\boldsymbol{L}f)$ for any continuous function satisfying certain other conditions. The class $\boldsymbol{L}$ includes, *e. g.*, the pairs $(1, j^{-1})$ and $(sin, 1/(1+j^2))$.

Adherence to this typography saves many parentheses and a great deal of verbiage.

(4) In the traditional literature, the function j is referred to as the function x (which assumes the value x for any number x).

(5) The vacuous subclass of a function is the vacuous function.

(6) In some earlier publication I had confined Π to 1-1 mappings of a subset of Dom u on a subset of Dom w.

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