REFERENCES

[1] A. Białynicki-Birula, *Remarks on quasi-Boolean algebras*, Bulletin de l'Académie Polonaise des Sciences, Classe III, 5 (1957), p. 615-619.

[2] — and H. Rasiowa, *On the representation of quasi-Boolean algebras*, ibidem, p. 259-261.

[3] A. Heyting, *Die formalen Regeln der intuitionistischen Logik*, Sitzungsberichte der Preussischen Akademie der Wissenschaften 1930, p. 42-56.

[4] J. C. C. McKinsey and A. Tarski, *On closed elements in closure algebras*, Annals of Mathematics 47 (1946), p. 122-162.

[5] — *Some theorems about the sentential calculi of Lewis and Heyting*, Journal of Symbolic Logic 13 (1948), p. 1-15.

[6] А. А. Марков, *Конструктивная логика*, Успехи Математических Наук 5.3 (1950), p. 187-188.

[7] D. Nelson, *Constructible falsity*, Journal of Symbolic Logic 14 (1949), p. 16-26.

[8] H. Rasiowa, *Algebraic models of axiomatic theories*, Fundamenta Mathematicae 41 (1954), p. 291-310.

[9] — *$\mathfrak{N}$-lattices and constructive logic with strong negation*, ibidem (to appear).

[10] — and R. Sikorski, *Algebraic treatment of the notion of satisfiability*, ibidem 40 (1953), p. 62-95.

[11] — *On existential theorems in non-classical functional calculi*, ibidem 41 (1954), p. 21-28.

[12] Н. Н. Воробьев, *Конструктивное исчисление высказываний с сильным отрицанием*, Доклады Академии Наук СССР 85 (1952), p. 456-468.

[13] — *Проблема выводимости в конструктивном исчислении высказываний с сильным отрицанием*, ibidem, p 689-692.

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COLLOQUIUM MATHEMATICUM

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ON CONVERGENCE OF MAPPINGS

BY

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1. Introduction. The idea of a mapping, that is a single valued continuous transformation, as an extension of the function concept lies deep in the history of topology. Indeed it is closely interlaced with the very beginnings of topology, as is apparent to any student of complex function theory with its strong emphasis on mappings generated by differentiable functions. In more recent times, however, one of the most powerful and stimulating influences in the development of topology and its applications has been the method of generation of mappings by decomposition of the domain space into disjoint closed sets, together with the dual operation of generating a decomposition of a domain space by means of a given mapping defined on that space. The early recognition by Kuratowski [1] of the equivalence of these operations in an appropriate setting and his formulation of some of the then current work on upper semi-continuous decompositions in terms of mappings surely represents a distinct landmark in the development of Analytic Topology and has lead to major advances in this area of mathematical work. It is a privilege and a pleasure, therefore, for the author to dedicate this paper to his long-time friend and colleague Casimir Kuratowski on the occasion of the 40th anniversary of his first mathematical publication. The author's mathematical life and work have been immeasurably stimulated and enriched through personal and professional association with this great mathematician and by his masterful and exceptional skill in topological writing and exposition.

We shall be concerned in this paper with sequences of mappings from one locally compact separable metric space to another*. Conditions for the almost uniform convergence of such sequences having some applicability in the case of function sequences will be studied. The existence

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of a limit mapping toward which the sequence converges in a certain sense will be assumed. Emphasis is centered on the case of mappings with non-compact domain and range spaces. The mappings themselves are usually assumed to have some elements of compactness. A mapping $f: X \to Y$ is *compact* provided $f^{-1}(K)$ is compact for every compact set K in Y or, equivalently, provided the mapping is closed and has compact point inverses $f^{-1}(y)$ for each $y \in Y$.

In general, standard notation and terminology will be employed. For each point x in a space X and each real number $r > 0$, $V_r(x)$ denotes the spherical neighbourhood of x of radius r. Also for any open set R in X, $\mathrm{Fr}(R)$ denotes the boundary or frontier of R. A sequence of sets A_n is said to converge *0-regularly* to the set A provided that A_n converges to A and for any $\varepsilon > 0$ a $\delta > 0$ and an integer N exist such that if $n > N$ any pair of points $x, y \in A_n$ with $\varrho(x, y) < \delta$ lie together in a connected set in A_n of diameter $< \varepsilon$.

2. Mapping types. Two kinds of mappings will be considered, *monotone* and *quasi-open*. A mapping $f: X \to Y$ is *monotone* provided $f^{-1}(y)$ is a continuum (i. e., compact and connected) for each $y \in Y$ and is *weakly monotone* provided each point inverse $f^{-1}(y)$ is connected. Also a mapping $f: X \to Y$ is (strongly) *quasi-open* provided that for any $y \in Y$ and any open set U in X which contains a compact component of $f^{-1}(y)$, y is interior to $f(U)$.

For the case of a mapping of one real line into another it is readily verified that *the properties of weak monotoneity and quasi-openness are equivalent and each is equivalent to the property of univariance when the mapping is regarded as a real valued function of a real variable.* This is no longer true for mappings of intervals onto intervals, because every open mapping is quasi-open and the interval is readily mapped onto itself by an open mapping which is not monotone. It may also be remarked that every (strictly) monotone mapping of one line into another is necessarily compact.

The mapping of a region R of the complex plane generated by a function differentiable in R is open and hence quasi-open. In this situation of a mapping generated by a function of a complex variable, quasi-openness is equivalent [2] to a minimum modulus property of the function. For our present purposes the following characterization of quassi-openness in a broader setting will be particularly useful. Let X and Y be locally connected generalized continua, i. e. locally compact connected and locally connected separable metric spaces. *A mapping $f: X \to Y$ is quasi--open if and only if the relation*

$$\mathrm{Fr}[f(R)] \subset f[\mathrm{Fr}(R)] \tag{*}$$

holds for every conditionally compact region R in X.

The word "region" here means "connected open set". If this word is replaced by "open set" the resulting characterization is valid in case X and Y are arbitrary *generalized continua* or, indeed, locally compact separable metric spaces.

3. Conditions. Example. It is well known, of course, that simple convergence of a sequence of mappings to a mapping does not by itself imply any type of uniformity of convergence in general. The classical theorems of Osgood and Vitali do give such conclusions for sequences of analytic functions. It is readily seen, however, that even the strongest topological restrictions on the separate mappings in the sequence is unlikely to have uniform convergence implications for the sequence as a whole. This may be illustrated by the

EXAMPLE. *There exists a sequence of homeomorphisms of the plane onto itself which converges everywhere to the identity but where the convergence fails to be almost uniform.*

To see this let C be the circle $x^2+y^2 = 1$. For each $n > 0$ let s_n be the arc of the upper semi-circle of C joining the points with abscissas $1/n$ and $1/(n+1)$, let c_n be the chord of C subtending s_n and let S_n and T_n be isosceles triangles of altitudes $\frac{1}{2}$ and 1 respectively standing on c_n as base and extending outside C. For each $n > 0$ let h_n be the homeomorphism of the plane onto itself which

(i) is the identity on and outside T_n,

(ii) maps the interior of T_n onto itself by sending s_n onto the union of the two sides of S_n outside C, the segment of the interior of C cut off by c_n and s_n going onto the interior of S_n and the part of the interior of T_n outside C going onto the part of the interior of T_n outside S_n. It is then readily verified that this sequence of homeomorphisms $h_1, h_2, h_3, \ldots$ meets all conditions required of our example. The sequence converges at each point to the identity; but the convergence fails to be almost uniform because for any n there exists a point p of s_n whose h_n image is the vertex of S_n and hence at a distance $\geqslant \frac{1}{4}$ from p.

Thus even for sequences of homeomorphisms extra conditions seem to be essential in order to secure uniform convergence conclusions about the sequence. It may be noted that in the example just described, any small circular neighbourhood U about the point $(0, 1)$ has the property that the limit of the images $h_n[\mathrm{Fr}(U)]$ of its boundary is a different set from the image $h[\mathrm{Fr}(U)]$ of its boundary under the limit h of the sequence. This suggests consideration of the associativity condition

$$\lim_{n\to\infty}[f_n(C)] = [\lim_{n\to\infty} f_n](C) \tag{+}$$

for sequences of mappings $f_n: X \to Y$ and subsets C of X. It will be shown in the next sections that a modified form of this condition applied to boundaries of small neighbourhoods of points does indeed yield conclusions of the desired sort on almost uniform convergence of the sequence.

4. Monotone mappings. In order to simplify the form of the condition needed we first give the

DEFINITION. If A_n is a sequence of sets in a metric space, we define

$$\limsup A_n \subset_s A,$$

read "limit superior of A_n strictly contained in A", to mean that for every $\varepsilon > 0$ almost all the sets A_n lie entirely in $V_\varepsilon(A)$. In a compact space this clearly is equivalent to the, in general weaker, statement "$\limsup A_n \subset A$".

THEOREM. *Let the closed generalized continua X, Y and Y' in a locally compact separable metric space be such that $Y' \subset Y$ and for each $y \in Y$, Y' intersects each component of $Y - y$. Let the sequence of compact monotone mappings $f_n(X) = Y_n$ and the mapping $f(X) = Y'$ satisfy*

(a) *the sets Y_n converge 0-regularly to Y and*

(b) *for each $x \in X$ and $\varepsilon > 0$ there exists an ε-neighborhood U of x with boundary C such that $\limsup f_n(C) \subset_s f(C)$.*

Then $[f_n(x)]$ converges almost uniformly to $f(x)$ on X and if each $f^{-1}(y)$, $y \in Y'$, has a non-empty compact component, f is compact and monotone.

Proof. We first establish the almost uniform convergence. Suppose, on the contrary, that on some compact set A in X, f_n does not converge uniformly to f. Then for some $\alpha > 0$ and each n there exists an $x_n \in A$ and an integer $k_n \geqslant n$ such that if $z_n = f_{k_n}(x_n)$ and $y_n = f(x_n)$ we have

(i) $$\varrho(y_n, z_n) > \alpha.$$

We may and do suppose α chosen so that $\overline{V_\alpha(y)}$ is compact. By taking a subsequence if necessary we may suppose $x_n \to x \in A$ and from this we have $y_n \to y = f(x)$ by continuity of f.

By regular convergence of Y_n to Y it follows that Y is locally connected. Thus if F_1 is the component of $Y \cdot V_\alpha(y)$ containing y, F_1 is a locally connected region in Y and there exists a positive number $\beta < \alpha/2$ such that

(ii) $$\overline{V_{2\beta}(y)} \cdot Y \subset F_1.$$

Again by local connectedness of F_1 there exists a positive number $\sigma < \beta/2$ such that if there are exactly r components $W_1, W_2, \ldots, W_r$ of $Y - y$ not contained wholly in $V_\beta(y)$, then each W_i contains a point b_i of $Y' \cdot [Y - V_\sigma(y)]$. For each $i \leqslant r$, W_i contains a finite number of components of $F_1 - y$. Let a point in each such component of $F_1 - y$ in W_i be joined to b_i by a simple arc in W_i. We then add this finite set of arcs to F_1 for each $i \leqslant r$ and let F_2 be the resulting set. Then $\overline{F}_2 - y$ has exactly r components $K_1, K_2, \ldots, K_r$ which do not lie entirely in $V_\beta(y)$ and we choose the notation so that $K_i \subset W_i$ for each $i \leqslant r$. Now let $\gamma < \sigma/2$ be chosen so that $\overline{F}_2 - \overline{F}_2 \cdot V_\sigma(y)$ is contained in the union of r components $M_1, M_2, \ldots, M_r$ of $\overline{F}_2 - \overline{F}_2 \cdot V_{2\gamma}(y)$ with notation so chosen that, for each $i \leqslant r$,

(iii) $$M_i \subset K_i \subset W_i.$$

By regular convergence of Y_n to Y there exists an integer N_1 and a $\delta > 0$ such that if $y', y'' \in Y_n$ with $n > N_1$ and $\varrho(y', y'') < \delta$, then $y' + y'' \subset G \subset Y_n$ where G is connected and $\delta(G) < \gamma/3$. Then by convergence of Y_n to Y, there exists an $N_2 > N_1$ such that if $n > N_2$ any point of $\overline{F}_2$ is at a distance $< \delta/6$ from some point of Y_n and any point of $Y_n \cdot \overline{V}_\beta(y)$ is at a distance $< \delta/6$ from some point of Y.

Now let U be an open set in X containing x such that

(iv) $$f(\overline{U}) \subset V_\gamma(y)$$

and such that $\limsup f_n(C) \subset_s f(C)$ where C is the boundary of U. There exists an integer $N_3 > N_2$ such that if $n > N_3$,

(v) $$f_n(C) \subset V_\gamma(y).$$

We shall show next that there exists an integer $N_4 > N_3$ such that if $n > N_4$, $f_n(X - \overline{U})$ intersects each component of $Y_n - Y_n \cdot V_\gamma(y)$ which does not lie wholly in $V_\beta(y)$. For each $i \leqslant r$ let z_i be a point of $f^{-1}(b_i)$. Then $z_i \in X - \overline{U}$ for each i by (iv), since $\varrho(b_i, y) \geqslant \sigma > 2\gamma$. For each $i \leqslant r$, let V_i be an open set about z_i with boundary C_i such that $\overline{V}_i \subset X - \overline{U}$, $f(\overline{V}_i) \subset V_\delta(b_i)$, and $\limsup f_n(C_i) \subset_s f(C_i)$. Then for each $i \leqslant r$ there exists an integer $N_4^i > N_3$ such that for $n > N_4^i$,

(vi) $$f_n(C_i) \subset V_{\delta/6}[f(C_i)] \subset V_{\delta/3}(b_i).$$

Let $N_4 = \max[N_4^i]$, let n be any integer $> N_4$ and let Q be any component of $Y_n - Y_n \cdot V_\gamma(y)$ which does not lie wholly in $V_\beta(y)$. There is a point p of Q on the boundary of $V_\beta(y)$ and, since $n > N_2$, p is at a distance $< \delta/6$ from some $b \in Y$.

Now by (ii), $b \in F_1 \subset F_2$ and b is not in $V_\sigma(y)$ since $\beta > 2\sigma$. Hence b lies in M_j for some $j \leqslant r$. Let $b = x_1, x_2, \ldots, x_k = b_j$ be a $\delta/6$-chain in M_j. Then if for each s, $1 < s \leqslant k$, y_s is a point of Y_n with $\varrho(x_s, y_s) < \delta/6$ and $y_k \in f_n(C_j)$, then $p = y_1, y_2, \ldots, y_{k-1}, y_k$ is a δ-chain in Y_n from p to y_k. For each $s < k$ there exists a connected subset E_s of Y_n containing $y_s + y_{s+1}$ and of diameter $< \gamma/3$. The union E of these sets E_s, $1 \leqslant s < k$, is a connected subset of Y_n containing $p + y_k$; and

$E \subset V_\gamma(M_j)$ since $\delta \leqslant \gamma/3$. Thus $E \subset Y_n - Y_n \cdot V_\gamma(y)$ so that $E \subset Q$. Hence Q contains the point y_k of $f_n(X-\overline{U})$ as was to be shown.

Now if n is any integer $> N_4$, we must have $f_n(U) \subset V_\beta(y)$. For suppose there is a point w of $f_n(U)$ in $Y_n - Y_n \cdot V_\beta(y)$. Then w lies in a component Q of $Y_n - Y_n \cdot V_\gamma(y)$ which is not contained in $V_\beta(y)$ and thus Q must contain a point z of $f_n(X-\overline{U})$ as shown above. Thus $f_n^{-1}(Q)$ intersects both U and $X-\overline{U}$. Since f_n is compact and monotone, $f_n^{-1}(Q)$ is connected and thus it must intersect C, the boundary of U. However, this is impossible because by (v), $f_n(C) \subset V_\gamma(y)$, as $n > N_4 > N_3$, so that $f_n(C) \cdot Q = 0$. Accordingly $f_n(U) \subset V_\beta(y)$.

Now for n sufficiently large, say $n > N_5 > N_4$, we have $x_n \in U$ and $y_n = f(x_n) \in V_\beta(y)$. However, this gives $z_n \in f_{k_n}(U) \subset V_\beta(y)$ so that $\varrho(y_n, z_n) < 2\beta < \alpha$, contrary to (i). Thus the supposition of non-uniform convergence on A leads to a contradiction.

To prove the final statement, suppose $f^{-1}(y)$ has a non-empty compact component for each $y \in Y$. Then for any $y \in Y$, if K is such a compact component of $f^{-1}(y)$, σ is any positive number and U is any conditionally compact open set about K with $\overline{U} \subset V_\sigma(K)$ and with boundary C not intersecting $f^{-1}(y)$, we can choose an open subset R of Y about y as follows. By regular convergence of Y_n to Y there exists a $\delta > 0$ and an N such that if $n > N$ and $y', y'' \in Y_n$ with $\varrho(y', y'') < \delta$, then $y' + y'' \subset Q \subset Y_n$ where Q is connected and of diameter $< \varepsilon = \frac{1}{3}\varrho[y, f(C)]$. We may, and do, suppose N chosen also so that $f_n(C) \subset V_\varepsilon[f(C)]$ for any $n > N$. Then if R denotes the set $Y \cdot Y_{\delta/2}(y)$ we must have

$$f^{-1}(R) \subset U. \tag{vii}$$

For if there existed a $z \in f^{-1}(R) \cdot (X-U)$, for $p \in K$ and n sufficiently large and $> N$ we would have $f_n(p) + f_n(z) \subset V_{\delta/2}(y)$ so that $f_n(p) + f_n(z) \subset Q \subset Y_n$, where Q is connected and of diameter $< \varepsilon$. Thus $Q \cdot f_n(C) = 0$ so that $f_n^{-1}(Q) \cdot C = 0$. This is impossible as $f_n^{-1}(Q)$ is connected and contains $p+z$. This proves (vii) and our result follows readily from this. For if $H \subset Y$ is compact, it can be covered by a finite collection of such sets R so that $f^{-1}(H)$ lies in a conditionally compact subset of X and thus is compact. Whence f is compact. Also, f is monotone, because $f^{-1}(y)$ must reduce to K since $f^{-1}(y) \subset U \subset V_\sigma(K)$ and σ is any positive number.

5. Notes. (1). As an immediate consequence we have the result that *any sequence of real continuous monotone non-decreasing functions $f_n(x)$ on the interval (a, b) which converges on an everywhere dense set including a and b to a function $f(x)$ continuous on (a, b) necessarily converges uniformly to $f(x)$ on (a, b)*[1]. For in this case the convergence at a and b together with the nature of the image sets ensures the regular convergence required in condition (a); and pointwise convergence at an everywhere dense set on (a, b) implies condition (b) immediately. For a discussion of further applications of this type reader is referred to the author's paper [3].

(2) The homeomorphisms: $f_n(x) = x/n$, $0 \leqslant x < 1$, $f_n(x) = x$, $-1 \leqslant x \leqslant 0$, converge almost uniformly to the mapping: $f(x) = 0$, $0 \leqslant x < 1$, $f(x) = x$, $-1 \leqslant x \leqslant 0$, and also Y_n converges 0-regularly to Y. However, f is neither compact nor monotone.

(3) Examples are readily constructed showing that all conditions in the theorem may be satisfied and yet Y' be different from Y.

6. Quasi-open mappings. Let X and Y be locally connected generalized continua and suppose that for each $y \in Y$ each component of $Y - y$ is non-compact. We then have the

THEOREM. *Let $f_n\colon X \to Y$ be a sequence of quasi-open mappings and suppose there exists a mapping $f\colon X \to Y$ such that for each $x \in X$ and $\varepsilon > 0$ there is a conditionally compact region R containing x and lying in $V_\varepsilon[f^{-1}f(x)]$ on the boundary C of which the relation*

$$\limsup f_n(C) \subset_s f(C) \tag{++}$$

holds. Then $f_n(x)$ converges almost uniformly to $f(x)$ on X.

To prove this we first repeat the first paragraph of the proof given in § 4. Then let $\sigma = \alpha/2$ and let V be a neighborhood of y lying in $V_\sigma(y)$ and so chosen that $Y - V_\sigma(y)$ is contained in the union M of a finite number of components $M_1, M_2, \ldots, M_k$ of $Y - V$ and no two of the sets M_i lie together in the same component of $Y - y$. Then each M_i is necessarily non-compact because it contains all points in $Y - V_\sigma(y)$ of some component of $Y - y$.

From our hypothesis it follows by continuity of f that there exists a conditionally compact region R in X containing x and such that $f(\overline{R}) \subset V$ and such that (++) holds on the boundary C of R. Since $f(C) \subset V$, there exists an integer N such that $f_n(C) \subset V$ for $n > N$. However this implies that $f_n(R) \subset V_\sigma(y)$ for $n > N$. For if not we would have $f_n(R) \cdot M_i \neq 0$ for some $i \leqslant k$; and since $f_n(\overline{R})$ is compact and thus cannot contain all of M_i, we would have $M_i \cdot \mathrm{Fr}[f_n(R)] \neq 0$. This contradicts relation

[1] The theorem remains true even if we drop the assumption that the functions $f_n(x)$ are continuous; see M. Nosarzewska, *On uniform convergence in some classes of functions*, Fundamenta Mathematicae 39 (1952), p. 38-52, theorem I [Note of the Editors].

(*) in § 2 because $f_n[\mathrm{Fr}(R)] = f_n(C)$ and this set lies in V. Accordingly $f_n(R) \subset V_\sigma(y)$.

Now for n sufficiently large, say $n > N_1 > N$, we have $x_n \in R$ and $y_n = f(x_n) \in V_\sigma(y)$. However this gives $z_n \in f_{k_n}(R) \subset V_\sigma(y)$ so that $\varrho(y_n, z_n) < 2\sigma = a$, contrary to relation (i) (see first paragraph of the proof in § 4). Thus the supposition of non-uniform convergence on A leads to a contradiction.

7. Conclusion. Since for a real continuous function on the whole real axis, or on a connected open set of real numbers, monotoneity of the function is equivalent to quasi-openness of the mapping generated by the function, the theorem just proven gives at once the result: *any sequence of monotone non-increasing continuous real functions on the whole real axis which converges at an everywhere dense set to a function $f(x)$ which is continuous for all real x, necessarily converges almost uniformly to $f(x)$.*

It seems likely, however, that our theorem for quasi-open mappings may be of greater interest in connection with sequences of functions of a complex variable or of mappings on surfaces and other more complex spaces. The setting provided by a closed algebra of complex valued functions seems of special interest and it is proposed to study this in a later paper.

REFERENCES

[1] C. Kuratowski, *Sur les décompositions semi-continues d'espaces métriques compacts*, Fundamenta Mathematicae 11 (1928), p. 169-185.

[2] G. T. Whyburn, *Quasi-open mappings*, appearing in the volume of Revue de Mathématiques pures et appliquées of the Academia RPR, Institutul de Matematica Bucharest, dedicated to Professor S. Stoïlow.

[3] — *Uniform convergence for monotone mappings*, Proceedings of the National Academy of Sciences 43, 11 (1957).

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ON A CERTAIN DISTANCE OF SETS AND THE CORRESPONDING DISTANCE OF FUNCTIONS

BY

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It is well known that the measure of the symmetric difference of two sets can be considered as a distance of sets (so called *distance of Fréchet-Nikodym-Aronszajn*): $\varrho(A, B) = \mu(A \dot{-} B)$. This distance is a particular case of the distance in the space of Lebesgue integrable functions.

This paper is devoted to the study of another distance of sets, defined by the formula

$$\sigma(A, B) = \frac{\mu(A \dot{-} B)}{\mu(A+B)} = \frac{\varrho(A, B)}{\mu(A+B)},$$

and the corresponding distance of functions.

The distance σ seems to be useful in several practical applications and especially in some biological problems (see n° 3 and our paper on a systematical distance of biotopes [2]).

1. SETS

1.1. Metric ϱ. Let $(X, \boldsymbol{M}, \mu)$ be a σ-finite σ-measure space. Let us denote by $\boldsymbol{M}_0$ the class of all sets $A \in \boldsymbol{M}$ with $\mu(A) < \infty$, and by ϱ_μ the well-known distance of sets $A, B \in \boldsymbol{M}_0$:

$$\varrho_\mu(A, B) = \mu(A \dot{-} B),$$

where $A \dot{-} B$ denotes the symmetric difference of A and B.

The index μ will be omitted in this case and in other analogous ones, when no misunderstanding is possible.

Let us recall the fundamental properties of ϱ (see e. g. [1], p. 168 and 169):

(i) *$(\boldsymbol{M}_0, \varrho)$ is a metric space when we identify any two sets, the symmetric difference of which is of measure μ zero.*