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ON CONSTRUCTIBLE FALSITY IN THE CONSTRUCTIVE LOGIC WITH STRONG NEGATION

BY

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This article is a continuation of paper [9] of Rasiowa, in which an algebraic characterization of the system $\mathcal{S}$ of the constructive propositional calculus with strong negation was given. The terminology and the notation is here the same as in [9]. The knowledge of [9] is here assumed.

The idea of the above-mentioned constructive logic with strong negation is due to David Nelson, who introduced in paper [7] a new constructive interpretation for logical connectives of the number theory and characterized a formal system of the number theory satisfying this interpretation. An analogical system of the number theory was later investigated by Markov [6]. The system of the propositional calculus with strong negation was examined by Vorobiev [12] and [13].

Under Nelson's interpretation of logical connectives the strong negation of a conjunction $\sim(\alpha\cdot\beta)$ is valid in case when at least one of the formulas $\sim\alpha$, $\sim\beta$ is valid and a formula $\sim\prod_{x_k}\alpha(x_k)$ is valid if and only if there exists such an x_p that $\sim\alpha\binom{x_p}{x_k}$ is valid.

We deal in this paper with the above-mentioned system $\mathcal{S}$ and with the system $\mathcal{S}^*$ of the functional calculus based on $\mathcal{S}$. The algebraic characterization of $\mathcal{S}$ is here generalized on $\mathcal{S}^*$. Using algebraic and topological methods we prove that according to the idea of Nelson a formula $\sim(\alpha\cdot\beta)$ is provable in $\mathcal{S}$ or in $\mathcal{S}^*$ if and only if at least one of the formulas $\sim\alpha$, $\sim\beta$ is provable. Similarly, a formula $\sim\prod_{x_k}\alpha(x_k)$ is provable in $\mathcal{S}^*$ if and only if for some x_p the formula $\sim\left(\alpha\binom{x_p}{x_k}\right)$ is provable. The above mentioned theorems are equivalent to the theorems stating that a disjunction $\alpha+\beta$ is provable in $\mathcal{S}$ or in $\mathcal{S}^*$ if and only if at least one of the formulas α, β is provable and that a formula $\sum_{x_k}\alpha(x_k)$ is provable if and only if for some x_p the formula $\alpha\binom{x_p}{x_k}$ is provable. The decidability of formulas of $\mathcal{S}^*$ having the prenex normal form follows from the last theorem.

Moreover, the algebraic method of examination of the systems $\mathcal{S}$ and $\mathcal{S}^*$ permits us to show other theorems concerning provability in these systems. E. g. a formula without the sign of strong negation is provable in $\mathcal{S}$ or in $\mathcal{S}^*$ if and only if it is provable in the Heyting propositional calculus or in the Heyting functional calculus, respectively. This result concerning provability in $\mathcal{S}$ was first stated by Vorobiev.

Some other connections between provability in $\mathcal{S}$ and in the Heyting propositional calculus lead to the possibility of constructing simple functionally free $\mathcal{N}$-lattices of sets, e. g. it is proved that every $\mathcal{N}$-lattice connected with the Heyting algebra of all open subsets of an n-dimensional Euclidean space is functionally free.

We also give a very simple decision method for formulas of $\mathcal{S}$ having the form $\alpha \equiv \beta$ (strong equivalence) where α and β are formulas without the sign of implication and without the sign of intuitionistic negation.

§ 1. The constructive propositional calculus with strong negation. The construction of the system $\mathcal{S}$ of the *constructive propositional calculus with strong negation* can briefly be described as follows. The system $\mathcal{S}$ contains as primitive symbols the parentheses, the propositional variables $p_1, p_2, \ldots$, and the following connectives: the *disjunction* sign $+$, the *conjunction* sign $\cdot$, the *strong negation* sign $\sim$, the *implication* sign $\rightarrow$, the *negation* sign $\neg$. We assume that the notion of a formula is familiar. The propositional variables are formulas of order one. If α is a formula of order n then $\sim\alpha$ and $\neg\alpha$ are formulas of order $n+1$. If α and β are formulas whose orders are not greater than n and the order of at least one of them is n, then the formulas $\alpha+\beta$, $\alpha\cdot\beta$, $\alpha\rightarrow\beta$ are of the order $n+1$.

The set $\mathcal{A}$ of all *axioms* of the system $\mathcal{S}$ is composed of all formulas of the following forms:

(1) $\alpha\rightarrow(\beta\rightarrow\alpha)$, (2) $\big(\alpha\rightarrow(\beta\rightarrow\gamma)\big)\rightarrow\big((\alpha\rightarrow\beta)\rightarrow(\alpha\rightarrow\gamma)\big)$,
(3) $(\alpha\cdot\beta)\rightarrow\alpha$, (4) $(\alpha\cdot\beta)\rightarrow\beta$, (5) $(\gamma\rightarrow\alpha)\rightarrow\big((\gamma\rightarrow\beta)\rightarrow(\gamma\rightarrow(\alpha\cdot\beta))\big)$,
(6) $\alpha\rightarrow(\alpha+\beta)$, (7) $\beta\rightarrow(\alpha+\beta)$, (8) $(\alpha\rightarrow\gamma)\rightarrow\big((\beta\rightarrow\gamma)\rightarrow((\alpha+\beta)\rightarrow\gamma)\big)$,
(9) $(\alpha\rightarrow\neg\beta)\rightarrow(\beta\rightarrow\neg\alpha)$, (10) $\neg\alpha\rightarrow(\alpha\rightarrow\beta)$, (11) $\sim\alpha\rightarrow(\alpha\rightarrow\beta)$,
(12) $\sim(\alpha\rightarrow\beta)\leftrightarrow(\alpha\cdot\sim\beta)$, (13) $\sim(\alpha\cdot\beta)\leftrightarrow(\sim\alpha+\sim\beta)$,
(14) $\sim(\alpha+\beta)\leftrightarrow(\sim\alpha\cdot\sim\beta)$, (15) $\sim\neg\alpha\leftrightarrow\alpha$, (16) $\sim\sim\alpha\leftrightarrow\alpha$,

where α, β, γ are arbitrary formulas of $\mathcal{S}$ and $\alpha\leftrightarrow\beta$ is written instead of $(\alpha\rightarrow\beta)\cdot(\beta\rightarrow\alpha)$. In the sequel we shall often write $\alpha\equiv\beta$ instead of $(\alpha\leftrightarrow\beta)\cdot(\sim\alpha\leftrightarrow\sim\beta)$. A formula is said to be *provable in* $\mathcal{S}$ provided that it belongs to the smallest set $\mathcal{T}$ of formulas containing all the axioms and closed under the *modus ponens*, i. e. such that if $\alpha, \alpha\rightarrow\beta\in\mathcal{T}$, then $\beta\in\mathcal{T}$. In particular, every formula of $\mathcal{S}$ being a substitution of a formula provable in the Heyting propositional calculus is provable in $\mathcal{S}$, e. g.

(17) $\alpha\rightarrow\big(\beta\rightarrow(\alpha\cdot\beta)\big)$, (18) $(\alpha\rightarrow\beta)\rightarrow\big((\beta\rightarrow\gamma)\rightarrow(\alpha\rightarrow\gamma)\big)$,
(19) $\big(\alpha\rightarrow(\beta\rightarrow\gamma)\big)\rightarrow\big(\beta\rightarrow(\alpha\rightarrow\gamma)\big)$.

1.1. *For a formula $\alpha(p_{i_1}, \ldots, p_{i_n})$, where $p_{i_1}, \ldots, p_{i_n}$ are all propositional variables appearing in α, there exists a formula $\alpha^0(p_{i_1}, \ldots, p_{i_n}, p_{j_1}, \ldots, p_{j_n})$ containing at most the propositional variables $p_{i_1}, \ldots, p_{i_n}, p_{j_1}, \ldots, p_{j_n}$ and for every $k = 1, \ldots, n$ at least one of the variables p_{i_k}, p_{j_k} where $j_k \neq j_l$, for $k\neq l$, $j_k\neq i_l$, $k, l = 1, \ldots, n$, such that the sign $\sim$ of the strong negation does not appear in α^0, and the formula*

$$\alpha(p_{i_1}, \ldots, p_{i_n}) \leftrightarrow \alpha^0(p_{i_1}, \ldots, p_{i_n}, \sim p_{i_1}, \ldots, \sim p_{i_n})\in\mathcal{T},$$

where $\alpha^0(p_{i_1}, \ldots, p_{i_n}, \sim p_{i_1}, \ldots, \sim p_{i_n})$ is the formula obtained from α^0 by substitution $\sim p_{i_k}$ for p_{j_k}, $k = 1, \ldots, n$.

This last formula will also be denoted by $s(\alpha^0)$.

The easy proof by induction with respect to the order of a formula α using (3)-(8) and (12)-(19) is omitted.

Obviously, if $\alpha^0(p_{i_1}, \ldots, p_{i_n}, p_{j_1}, \ldots, p_{j_n})$ fulfils the condition of theorem 1.1 for the formula α, i. e. $\alpha\leftrightarrow s(\alpha^0)\in\mathcal{T}$, then the formula obtained from α^0 by substitution of p_{m_k} for p_{j_k}, $k = 1, \ldots, n$, where $m_k\neq i_l$ for $k, l = 1, \ldots, n$ and $m_k\neq m_l$ for $k\neq l$ satisfies the same condition.

It is easy to show that

1.2. *Let $\alpha_1(p_{i_1}, \ldots, p_{i_k}, \ldots, p_{i_{k+l}})$, $\alpha_2(p_{i_1}, \ldots, p_{i_k}, p_{i_{k+l+1}}, \ldots, p_{i_{k+l+m}})$, where $i_r\neq i_s$ for $r\neq s$ be any formulas of $\mathcal{S}$. Let $\alpha_1^0(p_{i_1}, \ldots, p_{i_{k+l}}, p_{j_1}, \ldots, p_{j_{k+l}})$ and $\alpha_2^0(p_{i_1}, \ldots, p_{i_k}, p_{i_{k+l+1}}, \ldots, p_{i_{k+l+m}}, p_{j_1}, \ldots, p_{j_k}, p_{j_{k+l+1}}, \ldots, p_{j_{k+l+m}})$ where $j_r\neq j_s$ for $r\neq s$ and $j_r\neq i_s$, $r, s = 1, \ldots, k+l+m$ be the formulas satisfying the conditions of theorem 1.1 for α_1 and α_2, respectively. Then the formula $s(\alpha_1^0+\alpha_2^0)\leftrightarrow(\alpha_1+\alpha_2)$ is provable in $\mathcal{S}$, i. e. $\alpha_1^0+\alpha_2^0$ fulfils the condition of theorem 1.1 for the formula $\alpha_1+\alpha_2$.*

§ 2. The constructive functional calculus with strong negation. Let I_0 denote always the set of all positive integers. The system $\mathcal{S}^*$ of the *functional calculus* based on the constructive propositional calculus with strong negation contains as primitive symbols the parentheses, the individual variables $x_1, x_2, \ldots$, the functional variables $F_1, F_2, \ldots (k\in I_0)$ where $\nu(k)$ for $k\in I_0$ is the number of arguments of F_k, all connectives appearing in $\mathcal{S}$, and quantifiers $\sum_{x_i}$, $\prod_{x_i}$ where $i\in I_0$. We assume that the notions of formula and of a free and bound occurrence of an individual variable in a formula are familiar. The set $\mathcal{A}^*$ of all axioms of $\mathcal{S}^*$ consists of all formulas of the forms (1)-(16) given in § 1, where α, β, γ are arbitrary formulas of $\mathcal{S}^*$ and $\alpha\leftrightarrow\beta$ is written instead of $(\alpha\rightarrow\beta)\cdot(\beta\rightarrow\alpha)$. The set $\mathcal{T}^*$ of all provable formulas is the least set of formulas fulfilling the following conditions: (i) $\mathcal{T}^*$ contains all the axioms, (ii) if $\alpha, \alpha\rightarrow\beta\in\mathcal{T}^*$, then $\beta\in\mathcal{T}^*$; (iii) if $\alpha\in\mathcal{T}^*$ and β is obtained from α by the admissible replacement of all free occurrences of x_i by x_k, then $\beta \in \mathcal{T}^*$; (iv) if $\alpha \to \prod_{x_i} \beta \in \mathcal{T}^*$ then $\alpha \to \beta \in \mathcal{T}^*$; if $\sum_{x_i} \alpha \to \beta \in \mathcal{T}^*$, then $\alpha \to \beta \in \mathcal{T}^*$; if there is no free occurrence of x_i in α and $\alpha \to \beta \in \mathcal{T}^*$, then $\alpha \to \prod_{x_i} \beta \in \mathcal{T}^*$; if there is no free occurrence of x_i in β and $\alpha \to \beta \in \mathcal{T}^*$, then $\sum_{x_i} \alpha \to \beta \in \mathcal{T}^*$; (v) if $\alpha \to \sim \sum_{x_i} \beta \in \mathcal{T}^*$, then $\alpha \to \sim\beta \in \mathcal{T}^*$; if $\sim \prod_{x_i} \alpha \to \beta \in \mathcal{T}^*$ then $\sim\alpha \to \beta \in \mathcal{T}^*$; if there is no free occurrence of x_i in α and $\alpha \to \sim\beta \in \mathcal{T}^*$, then $\alpha \to \sim \sum_{x_i} \beta \in \mathcal{T}^*$; if there is no free occurrence of x_i in β and $\sim\alpha \to \beta \in \mathcal{T}^*$, then $\sim \prod_{x_i} \alpha \to \beta \in \mathcal{T}^*$.

For instance, the following formulas are provable in $\mathcal{S}^*$.

(17*) $\alpha \to \sum_{x_i} \alpha$, (18*) $\sim \sum_{x_i} \alpha \to \sim\alpha$, (19*) $\prod_{x_i} \alpha \to \alpha$, (20*) $\sim\alpha \to \sim \prod_{x_i} \alpha$, (21*) $\sum_{x_i} \sim\alpha \to \sim \prod_{x_i} \alpha$, (22*) $\sim \prod_{x_i} \alpha \to \sum_{x_i} \sim\alpha$, (23*) $\prod_{x_i} \sim\alpha \to \sim \sum_{x_i} \alpha$, (24*) $\sim \sum_{x_i} \alpha \to \prod_{x_i} \sim\alpha$, (25*) $\prod_{x_i} (\alpha \to \beta) \to (\prod_{x_i} \alpha \to \prod_{x_i} \beta)$; (26*) $\prod_{x_i} (\alpha \to \beta) \to (\sum_{x_i} \alpha \to \sum_{x_i} \beta)$, (27*) $\prod_{x_i} (\alpha \cdot \beta) \leftrightarrow (\prod_{x_i} \alpha \cdot \prod_{x_i} \beta)$, (28*) $\sum_{x_i} (\alpha + \beta) \leftrightarrow (\sum_{x_i} \alpha + \sum_{x_i} \beta)$, (29*) $(\prod_{x_i} \alpha + \prod_{x_i} \beta) \to \prod_{x_i} (\alpha + \beta)$, (30*) $\sum_{x_i} (\alpha \cdot \beta) \to \sum_{x_i} \alpha \cdot \sum_{x_i} \beta$.

§ 3. $\mathcal{N}$-lattices. Let $\boldsymbol{B} = \langle B, +, \cdot, \sim \rangle$ be a *quasi-Boolean algebra*, (see [2]), i. e. a distributive lattice with the unit element e and the zero element O, such that the following conditions are satisfied for the operation $\sim$ of the quasi-complement:

$$\sim(\sim a) = a, \qquad \sim(a \cdot b) = \sim a + \sim b \quad \text{for any } a, b \in B.$$

Clearly, every Boolean algebra is a quasi-Boolean algebra.

Let $\mathcal{X}$ be a non-empty set and let g be an involution of $\mathcal{X}$, i. e. a one-to-one mapping of $\mathcal{X}$ onto $\mathcal{X}$ such that

$$g(g(x)) = x \quad \text{for every} \quad x \in \mathcal{X}.$$

Setting

(20) $$\sim X = \mathcal{X} - g(X) \quad \text{for every} \quad X \subset \mathcal{X}$$

it is easy to verify that every family of subsets of $\mathcal{X}$ which contains X and is closed under thus defined operation $\sim$ of the quasi-complement, as well as under the set-theoretical operations of sum and product, is a quasi-Boolean algebra. It will be called a *quasi-field* of sets. It is known (see [2]) that

3.1. *For every quasi-Boolean algebra $\boldsymbol{B}$, there exist a set $\mathcal{X}$ and an involution g of $\mathcal{X}$ such that $\boldsymbol{B}$ is isomorphic with a subalgebra of the quasi-field of all subsets of $\mathcal{X}$. More precisely, the set $\mathcal{X}$ consists of all prime filters of $\boldsymbol{B}$, and the involution g is defined as follows:*

(21) $$g(\mathfrak{q}) = B - \tilde{\mathfrak{q}},$$

where $\tilde{\mathfrak{q}}$ is the set of all elements $\sim b$ such that $b \in \mathfrak{q}$, and the isomorphism h of $\boldsymbol{B}$ into the quasi-field of all subsets of $\mathcal{X}$ is given by the equality

(22) $$h(a) = \mathop{E}_{\mathfrak{q} \in \mathcal{X}} (a \in \mathfrak{q}).$$

An algebra $\boldsymbol{A} = \langle A, e, +, \cdot, \sim, \to, \neg \rangle$ is said to be an *$\mathcal{N}$-lattice* (cf. [9]) provided that

(n_1) A is quasi-ordered by the relation $<$ defined as follows:

$$a < b \text{ if and only if } a \to b = e,$$

i. e. the relation $<$ is reflexive and transitive;

(n_2) the abstract algebra $\langle A, +, \cdot, \sim \rangle$ is a quasi-Boolean algebra with the unit element e and the zero element $O = \sim e$; the relation $\subset$ defined by the equivalence

$$a \subset b \text{ if and only if } a < b \text{ and } \sim b < \sim a$$

is the partial ordering relation of this lattice;

(n_3) $a < c$ and $b < c$ imply $a + b < c$,

(n_4) $c < a$ and $c < b$ imply $c < a \cdot b$,

(n_5) $\sim(a \to b) < (a \cdot \sim b)$,

(n_6) $(a \cdot \sim b) < \sim(a \to b)$,

(n_7) $a < \sim \neg a$,

(n_8) $\sim \neg a < a$,

(n_9) $a \cdot \sim a < b$,

(n_{10}) $a < b \to c$ if and only if $a \cdot b < c$,

(n_{11}) $\neg a = a \to O$.

Let $\mathcal{X}_1$ be a topological space in which Int is the operation of interior, and let $\boldsymbol{H}(\mathcal{X}_1)$ be a Heyting algebra (¹) of open subsets of $\mathcal{X}_1$, constituting the class of neighbourhoods of $\mathcal{X}_1$. Let $\to_1$ denote the operation of pseudocodifference in $\boldsymbol{H}(\mathcal{X}_1)$ and let $\neg_1$ denote the operation of pseudocomplement in $\boldsymbol{H}(\mathcal{X}_1)$, i. e.,

(23) $$\begin{cases} X \to_1 Y = \operatorname{Int}((\mathcal{X}_1 - X) + Y) & \text{for any } X, Y \in \boldsymbol{H}(\mathcal{X}_1), \\ \neg_1 X = \operatorname{Int}(\mathcal{X}_1 - X). \end{cases}$$

Let $\mathcal{X}_2$ be a set of the same cardinal as $\mathcal{X}_1$. Let f be a one-to-one mapping of $\mathcal{X}_1$ onto $\mathcal{X}_2$ such that $f(x) = x$ for $x \in \mathcal{X}_1 \cdot \mathcal{X}_2$ and such that

(24) $$\mathcal{X}_1 \cdot \mathcal{X}_2 \subset \bigcap_{X \in \boldsymbol{H}(\mathcal{X}_1)} \big(f(\operatorname{Int}(\mathcal{X}_1 - X)) + X\big).$$

We shall set $\mathcal{X} = \mathcal{X}_1 + f(\mathcal{X}_1) = \mathcal{X}_1 + \mathcal{X}_2$. Then the mapping g

(¹) For the notion of the Heyting algebra of sets see e. g. [9].

of $\mathcal{X}$ onto $\mathcal{X}$ defined as

$$g(x) = \begin{cases} f(x) & \text{for } x \in \mathcal{X}_1, \\ f^{-1}(x) & \text{for } x \in \mathcal{X}_2 \end{cases}$$

is an involution of $\mathcal{X}$.

Let $\boldsymbol{B}(\mathcal{X})$ be the class of all subsets of $\mathcal{X}$ defined as follows: a subset $X \subset \mathcal{X}$ belongs to $\boldsymbol{B}(\mathcal{X})$ if and only if it fulfils the conditions

(i) $X \cdot \mathcal{X}_1 \in \boldsymbol{H}(\mathcal{X}_1)$,

(ii) there exists $Y \in \boldsymbol{H}(\mathcal{X}_1)$ such that $X \cdot \mathcal{X}_2 = \mathcal{X}_2 - g(Y)$,

(iii) $X \cdot \mathcal{X}_1 \subset g(X) \cdot \mathcal{X}_1$.

Then $\boldsymbol{B}(\mathcal{X})$ is a $\mathcal{N}$-lattice under the set-theoretical operations of sum and product, the operation $\sim$ of quasi complement defined by (20) and under the operations $\to$ of the $\mathcal{N}$-codifference and $\neg$ of the $\mathcal{N}$-complement defined as follows:

(25) $$X \to Y = (X \cdot \mathcal{X}_1 \to_1 Y \cdot \mathcal{X}_1) + (\mathcal{X}_2 - g(X \cdot \mathcal{X}_1)) + Y\mathcal{X}_2,$$

$$\neg X = X \to \wedge \quad \text{(cf. [9])}.$$

Notice that

(26) $$X < Y \text{ if and only if } \mathcal{X}_1 X \subset \mathcal{X}_1 Y.$$

The $\mathcal{N}$-lattice $\boldsymbol{B}(\mathcal{X})$ will be called an *$\mathcal{N}$-lattice* of sets *connected* with the Heyting algebra $\boldsymbol{H}(\mathcal{X}_1)$.

Its every subalgebra $\boldsymbol{B}_1(\mathcal{X})$ will be called an *$\mathcal{N}$-lattice of sets.* Let $\boldsymbol{H}_1(\mathcal{X}_1)$ be the class of subsets of $\mathcal{X}_1$ of the form

$$X \cdot \mathcal{X}_1 \text{ where } X \in \boldsymbol{B}_1(\mathcal{X}).$$

Then (see [9], (iii)) $\boldsymbol{H}_1(\mathcal{X}_1)$ is a Heyting algebra of sets, being a subalgebra of the Heyting algebra $\boldsymbol{H}(\mathcal{X}_1)$. The algebra $\boldsymbol{H}_1(\mathcal{X}_1)$ will be called the *basic Heyting algebra of* $\boldsymbol{B}_1(\mathcal{X})$.

It follows immediately from the above construction of an $\mathcal{N}$-lattice $\boldsymbol{B}(\mathcal{X})$ that

3.2. *For every Heyting algebra $\boldsymbol{H}(\mathcal{X}_1)$ there exists an $\mathcal{N}$-lattice of sets $\boldsymbol{B}(\mathcal{X})$ such that $\boldsymbol{H}(\mathcal{X}_1)$ is its basic Heyting algebra.*

It has been proved (see [9], (3.8)) that

3.3. *For every $\mathcal{N}$-lattice $\boldsymbol{A}$ there exists an $\mathcal{N}$-lattice of sets $\boldsymbol{B}_1(\mathcal{X})$ isomorphic with $\boldsymbol{A}$.*

We shall say that a *quasi-Boolean algebra $\boldsymbol{B} = \langle B, +, \cdot, \sim\rangle$ can be extended to an $\mathcal{N}$-lattice* provided that there exists an $\mathcal{N}$-lattice $\boldsymbol{A} = \langle A, e, +, \cdot, \sim, \to, \neg\rangle$ such that $\boldsymbol{B}$ is isomorphic with a subalgebra of the quasi-Boolean algebra $\langle A, +, \cdot, \sim\rangle$.

3.4. *A quasi-Boolean algebra $\boldsymbol{B} = \langle B, +, \cdot, \sim\rangle$ can be extended to an $\mathcal{N}$-lattice if and only if the following condition is satisfied:*

(27) $$(a \cdot \sim a) + (b + \sim b) = (b + \sim b) \text{ for any } a, b \in B.$$

The necessity of (27) follows from the fact that in every $\mathcal{N}$-lattice this condition is satisfied (see $(n_9)(n_2)$ § 3).

To prove the sufficiency, let us suppose that $\boldsymbol{B}$ is a quasi-Boolean algebra fulfilling condition (27). First of all we shall show that

(28) *either $g(\mathfrak{q}) \subset \mathfrak{q}$ or $\mathfrak{q} \subset g(\mathfrak{q})$ for every prime filter $\mathfrak{q}$ of $\boldsymbol{B}$, where the mapping g is defined by equality* (21).

In fact let us suppose that *non* $\mathfrak{q} \subset g(\mathfrak{q})$. Hence there exists an a such that $a \in \mathfrak{q}$ and $a \notin g(\mathfrak{q})$. It follows from $a \notin g(\mathfrak{q})$ that $\sim a \in \mathfrak{q}$. Thus $a \cdot \sim a \in \mathfrak{q}$. We shall show that $g(\mathfrak{q}) \subset \mathfrak{q}$. Let us suppose that $b \in g(\mathfrak{q})$. It is easy to see that $\sim b \notin \mathfrak{q}$. But it follows from (27) that $a \cdot \sim a \subset b + \sim b$. Consequently, $b + \sim b \in \mathfrak{q}$. Since $\mathfrak{q}$ is a prime filter, and $\sim b \notin \mathfrak{q}$, we infer that $b \in \mathfrak{q}$.

Let $\mathcal{X}_1$ be the set of all prime filters such that $\mathfrak{q} \subset g(\mathfrak{q})$, and let $\mathcal{X}_2$ be the set of all prime filters $\mathfrak{q}$ for which the condition $g(\mathfrak{q}) \subset \mathfrak{q}$ is satisfied. Let $\mathcal{X}$ be the set of all prime filters of $\boldsymbol{B}$. Obviously we have

$$\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2.$$

Moreover, it is easy to verify that $g(\mathcal{X}_1) = \mathcal{X}_2$. We shall treat $\mathcal{X}_1$ as a topological space with the discrete topology, i.e. we set $\operatorname{Int} X = X$ for every $X \supset \mathcal{X}_1$. Let $\boldsymbol{H}(\mathcal{X}_1)$ be the Heyting algebra of all subsets of $\mathcal{X}_1$. Let $\boldsymbol{B}(\mathcal{X})$ be the class of all subsets of $\mathcal{X}$ fulfilling the condition $X \cdot \mathcal{X}_1 \subset g(X \cdot \mathcal{X}_2)$. Then it is easy to see that this class coincides with the class of all subsets of $\mathcal{X}$ satisfying conditions (i), (ii), (iii). Hence, $\boldsymbol{B}(\mathcal{X})$ is an *$\mathcal{N}$-lattice* of sets. Moreover the class of all elements of $\boldsymbol{B}(\mathcal{X})$ of the form $X = h(a)$, $a \in B$, where h is defined by (22), is a subalgebra of the quasi-Boolean algebra of all subsets of $\mathcal{X}$ (under the set-theoretical operations of sum and product and the quasi-complement defined by (20)) isomorphic with $\boldsymbol{B}$.

The following abstract algebra $\boldsymbol{C}_0 = \langle C_0, +, \cdot, \sim\rangle$ is an example of a quasi-Boolean algebra satisfying condition (27): $\langle C_0, +, \cdot\rangle$ is the three-elements lattice consisting of elements O, a, e, where O is the *zero* element and e is the *unit* element. The operation $\sim$ in $\boldsymbol{C}_0$ is defined as follows:

$$\sim O = e, \quad \sim e = O, \quad \sim a = a.$$

The quasi-Boolean algebra $\boldsymbol{C}_1 = \langle C_1, +, \cdot, \sim\rangle$ which consists of four elements O, a, b, e where O is its zero element, e is its unit element,

$a+b = e$, and in which the operation of quasi-complement is defined by means of the equalities

$$\sim a = a, \ \sim b = b, \ \sim 0 = e, \ \sim e = 0,$$

is an example of quasi-Boolean algebra not fulfilling condition (27).

It is known (see [1]) that every quasi-Boolean algebra can be represented as a subdirect union of two-element Boolean algebras, of quasi-Boolean algebras C_0 and of quasi-Boolean algebras C_1. It is easy to verify that if a quasi-Boolean algebra fulfils condition (27), then it can be represented as a subdirect union of quasi-Boolean algebras C_0 and of two-elements Boolean algebras.

Let $\mathfrak{K}$ be the class of all quasi-Boolean algebras satisfying condition (27).

A quasi-Boolean algebra $\boldsymbol{A}\in\mathfrak{K}$ is said to be *functionally free for the class* $\mathfrak{K}$ provided that the following condition is fulfilled: any two quasi-Boolean polynomials Φ, Ψ are identically equal in every algebra of class $\mathfrak{K}$ if and only if they are identically equal in $\boldsymbol{A}$.

It follows from the remark given above that

3.5. *The quasi-Boolean algebra C_0 is functionally free for class $\mathfrak{K}$.*

Let $\boldsymbol{B}(\mathcal{X})$ be an $\mathcal{N}$-lattice of sets connected with a Heyting algebra $\boldsymbol{H}(\mathcal{X}_1)$ and let the operation $\sim$ of quasi-complement in $\boldsymbol{B}(\mathcal{X})$ be determined by the involution g. Now let an $\mathcal{N}$-lattice $\boldsymbol{B}_1(\mathcal{X})$ be a subalgebra of $\boldsymbol{B}(\mathcal{X})$.

Let us set $\mathcal{X}_1^0 = \mathcal{X}_1+\{x_1\}$ and $\mathcal{X}_2^0 = g(\mathcal{X}_1)+\{x_2\}$ where $x_1, x_2 \notin \mathcal{X}$ and $x_1 \neq x_2$. The set $\mathcal{X}_1^0$ will be considered as a topological space with the following topology: *the open subsets of $\mathcal{X}_1^0$ are all open subsets of $\mathcal{X}_1$ and the whole space $\mathcal{X}_1^0$*. Then the following theorem holds:

3.6. (i) *The class $\boldsymbol{H}(\mathcal{X}_1^0)$ consisting of all subsets $X\subset\mathcal{X}_1^0$ which belong to $\boldsymbol{H}(\mathcal{X}_1)$ and of $\mathcal{X}_1^0$ is a Heyting algebra of open subsets of $\mathcal{X}_1^0$.*

(ii) *The mapping*

$$g^0(x) = g(x) \text{ for } x\in\mathcal{X}, \qquad g^0(x_1) = x_2, \ g^0(x_2) = x_1$$

is an involution of $\mathcal{X}^0 = \mathcal{X}_1^0+\mathcal{X}_2^0$. Moreover, $g^0(x) = x$ for $x\in\mathcal{X}_1^0\cdot\mathcal{X}_2^0$.

(iii) $\mathcal{X}_1^0\cdot\mathcal{X}_2^0\subset \bigcap_{X\in \boldsymbol{H}(\mathcal{X}_1^0)} \big(g^0(\mathrm{Int}^0(\mathcal{X}_1^0-X))+X\big)$ *where* Int^0 *is the operation of interior in* $\mathcal{X}_1^0$.

(iv) *The class $\boldsymbol{B}(\mathcal{X}^0)$ of subsets of $\mathcal{X}^0 = \mathcal{X}_1^0+\mathcal{X}_2^0$ consisting of the empty set Λ, of $\mathcal{X}^0$ and of all sets $\mathcal{X} = G+\{x_2\}$ where $G\in\boldsymbol{B}(\mathcal{X})$ is an N-lattice of sets connected with $\boldsymbol{H}(\mathcal{X}_1^0)$.*

(v) *The class $\boldsymbol{B}_1(\mathcal{X}^0)$ consisting of the empty set Λ, of the set $\mathcal{X}^0$ and of all subsets $X = G+\{x_2\}$ where $G\in\boldsymbol{B}_1(\mathcal{X})$ is a subalgebra of $\boldsymbol{B}(\mathcal{X}^0)$.*

(vi) *The mapping $h(\mathcal{X}) = \mathcal{X}\cdot X$ for every $X\in\boldsymbol{B}_1(\mathcal{X}^0)$ is a homomorphism of $\boldsymbol{B}_1(\mathcal{X}^0)$ onto $\boldsymbol{B}_1(\mathcal{X})$.*

(vii) *If $X+Y = \mathcal{X}^0$ for $X, Y\in\boldsymbol{B}_1(\mathcal{X}^0)$, then either $X = \mathcal{X}^0$ or $Y = \mathcal{X}^0$. More generally, for a set $\{X_\mu\}_{\mu\in M}$ of elements of $\boldsymbol{B}_1(\mathcal{X}^0)$ for which there exists the sum in $\boldsymbol{B}_1(\mathcal{X}^0)$, if $\sum_{\mu\in M} X_\mu = \mathcal{X}^0$, then for some $\mu\in M$, $X_\mu = \mathcal{X}^0$.*

(i) is known, see e. g. [11]. (ii) obvious. Statement (iii) easily follows from the fact that $\mathcal{X}_1^0\cdot\mathcal{X}_2^0 = \mathcal{X}_1\cdot\mathcal{X}_2$ and $\mathcal{X}_1\cdot\mathcal{X}_2\subset\bigcap_{X\in\boldsymbol{H}(\mathcal{X}_1)}\big(g(\mathrm{Int}(\mathcal{X}_1-X))+X\big)$, since $\boldsymbol{B}(\mathcal{X})$ is connected with $\boldsymbol{H}(\mathcal{X}_1)$.

To prove (iv) it is sufficient to show that a subset X of $\mathcal{X}^0$ belongs to $\boldsymbol{B}(\mathcal{X}^0)$ if and only if it satisfies the conditions

(a) $\mathcal{X}_1^0\cdot X\in\boldsymbol{H}(\mathcal{X}_1^0)$,

(b) $\mathcal{X}_2^0\cdot X = \mathcal{X}_2^0-g^0(Y)$ where $Y\in\boldsymbol{H}(\mathcal{X}_1^0)$,

(c) $\mathcal{X}_1^0\cdot X\subset\mathcal{X}_1^0\cdot g^0(X)$.

It is easy to verify that the empty set Λ, $\mathcal{X}^0$ and every set $X = G+\{x_2\}$ where $G\in\boldsymbol{B}(\mathcal{X})$ satisfy conditions (a), (b), (c). Conversely, let us suppose that X fulfils all the conditions (a), (b), (c) and $X\neq\Lambda$, $X\neq\mathcal{X}^0$. We shall prove that $X = G+\{x_2\}$ where $G\in\boldsymbol{B}(\mathcal{X})$. First of all we shall show that $x_1\notin X$. In fact, if $x_1\in X$, then by (a) $X\cdot\mathcal{X}_1^0 = \mathcal{X}_1^0$. Hence by (c) $\mathcal{X}_2^0\subset X$. Thus $X = \mathcal{X}^0$, which contradicts our hypothesis. Now we shall show that $x_2\in X$. Indeed, $\sim X\neq\Lambda$, $\sim X\neq\mathcal{X}^0$ and $\sim X$ fulfils conditions (a), (b), (c). Consequently $x_1\notin\sim X$. Thus $x_1\in g^0(X)$. Since $x_2 = g^0(x_1)$ we obtain $x_2\in X$. Hence $X = G+\{x_2\}$ where $G\subset\mathcal{X}$. We shall prove that $G\in\boldsymbol{B}(\mathcal{X})$. It follows from (a) that $X\cdot\mathcal{X}_1^0 = G\cdot\mathcal{X}_1\in\boldsymbol{H}(\mathcal{X}_1)$. Condition (b) for the set X implies that $\mathcal{X}_2^0\cdot X = \mathcal{X}_2\cdot G+\{x_2\} = \mathcal{X}_2^0-g^0(Y)$ where $Y\in\boldsymbol{H}(\mathcal{X}_1)$. In fact, if $Y = \mathcal{X}_1^0$, then we should have $X = \mathcal{X}^0$, which contradicts our hypothesis. Consequently, $\mathcal{X}_2\cdot G = \mathcal{X}_2-g(Y)$, where $Y\in\boldsymbol{H}(\mathcal{X}_1)$. Since condition (c) is satisfied for X, we infer that $G\cdot\mathcal{X}_1\subset g(G)\cdot\mathcal{X}_1$. Hence $G\in\boldsymbol{B}(\mathcal{X})$. This completes the proof of (iv). To prove (v) let us notice that the following equalities hold in $\boldsymbol{B}(\mathcal{X}^0)$:

(29) $\sim\Lambda = \mathcal{X}^0$, $\sim\mathcal{X}^0 = \Lambda$, $\sim(G+\{x_2\}) = (\mathcal{X}-g(G))+\{x_2\}$,

hence, if $G\in\boldsymbol{B}_1(\mathcal{X})$, we infer that $\mathcal{X}-g(G)\in\boldsymbol{B}_1(\mathcal{X})$.

(30) $\Lambda+X = X$, $\mathcal{X}^0+X = \mathcal{X}^0$, $(G_1+\{x_2\})+(G_2+\{x_2\}) = (G_1+G_2)+\{X_2\}$.

(31) $\Lambda\cdot X = \Lambda$, $\mathcal{X}^0\cdot X = X$, $(G_1+\{x_2\})\cdot(G_2+\{x_2\}) = G_1\cdot G_2+\{x_2\}$.

(32) if $X\subset Y$, then $X\to Y = \mathcal{X}^0$

(33) $\mathcal{X}^0\to Y = Y$ if $Y\neq\mathcal{X}^0$

(34) $(G+\{x_2\})\to\Lambda = \begin{cases}\mathcal{X}^0 & \text{if } \mathcal{X}_1\cdot G = \Lambda\\ G_1+\{x_2\} & \text{if } \mathcal{X}_1\cdot G\neq\Lambda, \text{ where}\end{cases}$

$G_1 = G\to\Lambda$ in $\boldsymbol{B}(\mathcal{X})$,

$$(35)\qquad (G_1+\{x_2\})\to(G_2+\{x_2\}) = \begin{cases} \mathcal{X}^0 \text{ if } \mathcal{X}_1\cdot G_1\subset\mathcal{X}_1\cdot G_2 \\ G_3+\{x_2\} \text{ in the other case, where} \\ G_3 = G_1\to G_2 \text{ in } \boldsymbol{B}(\mathcal{X}).\end{cases}$$

It easily follows from (29)-(35) that $\boldsymbol{B}_1(\mathcal{X}^0)$ is a subalgebra of $\boldsymbol{B}(\mathcal{X}^0)$. Equalities (29)-(35) imply also that the mapping $h(X) = \mathcal{X}\cdot X$ for $X\in\boldsymbol{B}_1(\mathcal{X}^0)$ is a homomorphism of $\boldsymbol{B}_1(\mathcal{X}^0)$ onto $\boldsymbol{B}_1(\mathcal{X})$, i. e. that (vi) holds. Condition (vii) follows immediately from (v).

3.7. *Let $\boldsymbol{A}$ be an $\mathcal{N}$-lattice and let $\{a_\mu\}$, $\mu\in M$, be a set of elements of $\boldsymbol{A}$. Then:*

(i) *the existence of the sum $\sum_{\mu\in M} a_\mu$ (the product $\prod_{\mu\in M}\sim a_\mu$) implies the existence of the product $\prod_{\mu\in M}\sim a_\mu$ (the sum $\sum_{\mu\in M} a_\mu$) and the following equality holds:*

$$\sim\sum_{\mu\in M} a_\mu = \prod_{\mu\in M}\sim a_\mu.$$

(ii) *the existence of the product $\prod_{\mu\in M} a_\mu$ (the sum $\sum_{\mu\in M}\sim a_\mu$) implies the existence of the sum $\sum_{\mu\in M}\sim a_\mu$ (the product $\prod_{\mu\in M} a_\mu$) and the following equality holds:*

$$\sim\prod_{\mu\in M} a_\mu = \sum_{\mu\in M}\sim a_\mu.$$

The easy proof is omitted.

§ 4. Algebraic treatment of formulas of the constructive propositional calculus with strong negation. Every formula α of the system $\mathcal{S}$ of the constructive propositional calculus with strong negation will be interpreted as a polynomial $\alpha_{\boldsymbol{A}}$ of an $\mathcal{N}$-lattice $\boldsymbol{A} = \langle A, e, +, \cdot, \sim, \to, \neg\rangle$, by treating every propositional variable p_i, $i = 1, 2, \ldots$, as a variable running over the set A and every connective of $\mathcal{S}$ as the corresponding algebraic operation of $\boldsymbol{A}$. Let v be a valuation of all propositional variables of $\mathcal{S}$, i. e. a mapping of the set of all propositional variables into A. The value of the polynomial $\alpha_{\boldsymbol{A}}$ for the values of its variables fixed by v will be denoted by $\alpha_{\boldsymbol{A}v}$.

The following theorems were proved in [9]

4.1. *If α is a provable formula of $\mathcal{S}$, then $\alpha_{\boldsymbol{A}v} = e$ for every $\mathcal{N}$-lattice $\boldsymbol{A}$ and every valuation v in $\boldsymbol{A}$.*

4.2. *There exists an $\mathcal{N}$-lattice of sets $\boldsymbol{B}_0 = \langle B_0(\mathcal{X}_0), \mathcal{X}_0, +, \cdot, \sim, \to, \neg\rangle$ and a valuation v_0 in $\boldsymbol{B}_0$ such that for every formula α of $\mathcal{S}$, α is provable if and only if $\alpha_{\boldsymbol{B}_0 v_0} = \mathcal{X}_0$.*

Every formula α of $\mathcal{S}$ such that neither the sign $\to$ nor the sign $\neg$ appears in α may be interpreted in a familiar way as a quasi-Boolean polynomial $\alpha_{\boldsymbol{C}}$ of a quasi-Boolean algebra $\boldsymbol{C}$.

Let $\boldsymbol{C}_0$ be the quasi-Boolean algebra of theorem 3.5.

4.3. *Let α, β be arbitrary formulas of $\mathcal{S}$ without the sign $\to$, $\neg$. Then the formula $\alpha\equiv\beta$ is provable in $\mathcal{S}$ if and only if $\alpha_{\boldsymbol{C}_0} = \beta_{\boldsymbol{C}_0}$ identically*, i.e. $\alpha_{\boldsymbol{C}_0 v} = \beta_{\boldsymbol{C}_0 v}$ for every valuation v.

Since the signs $\to$ and $\neg$ do not appear in α and β we may interpret these formulas as quasi-Boolean polynomials $\alpha_{\boldsymbol{C}_0}$, $\beta_{\boldsymbol{C}_0}$ of $\boldsymbol{C}_0$. Let v be a valuation of propositional variables of $\mathcal{S}$ in $\boldsymbol{C}_0$. On account of 3.4 $\boldsymbol{C}_0$ can be extended to an $\mathcal{N}$-lattice $\boldsymbol{A} = \langle A, e, +, \cdot, \sim, \to, \neg\rangle$. Let h be an isomorphism of $\boldsymbol{C}_0$ into the quasi-Boolean algebra $\langle A, +, \cdot, \sim\rangle$. Let us set $u(p_i) = h(v(p_i))$ for $i = 1, 2, \ldots$ The formula $\alpha\equiv\beta$ being provable in $\mathcal{S}$ we infer from 4.1 and (n_1), (n_2) that $\alpha_{\boldsymbol{A}u} = \beta_{\boldsymbol{A}u}$. Since $v(p_i)\in\boldsymbol{C}_0$ we obtain $\alpha_{\boldsymbol{C}_0 v} = \beta_{\boldsymbol{C}_0 v}$.

Conversely, let us suppose that $\alpha\equiv\beta$ is not provable in $\mathcal{S}$. Then by 4.2 and (n_1), (n_2) it follows that $\alpha_{\boldsymbol{B}_0 v_0}\neq\beta_{\boldsymbol{B}_0 v_0}$. Since the signs $\to$, $\neg$ do not appear in α, β we can treat $\alpha_{\boldsymbol{B}_0}$ and $\beta_{\boldsymbol{B}_0}$ as quasi-Boolean polynomials of the quasi-Boolean algebra $\langle\boldsymbol{B}_0(\mathcal{X}_0), +, \cdot, \sim\rangle$. Consequently, by 3.5, there exists a valuation v in C_0 such that $\alpha_{\boldsymbol{C}_0 v}\neq\beta_{\boldsymbol{C}_0 v}$.

Let us suppose now that α is a formula of $\mathcal{S}$ such that the sign $\sim$ does not appear in α. Then α can be interpreted in the familiar way as a polynomial $\alpha_{\boldsymbol{H}(\mathcal{X}_1)}$ of a Heyting algebra $\boldsymbol{H}(\mathcal{X}_1)$ of sets. If u is a valuation of the propositional variables of $\mathcal{S}$ in $\boldsymbol{H}(\mathcal{X}_1)$, then the symbol $\alpha_{\boldsymbol{H}(\mathcal{X}_1)u}$ will denote the value of $\alpha_{\boldsymbol{H}(\mathcal{X}_1)}$ for the values of its arguments fixed by u.

Let $\boldsymbol{B}_1(\mathcal{X})$ be an $\mathcal{N}$-lattice of sets and $\boldsymbol{H}_1(\mathcal{X}_1)$ its basic Heyting algebra (see § 3).

4.4. *For every formula α of $\mathcal{S}$ without the sign $\sim$ and for every valuation v in $\boldsymbol{B}_1(\mathcal{X})$*

$$\alpha_{\boldsymbol{B}_1(\mathcal{X})v}\cdot\mathcal{X}_1 = \alpha_{\boldsymbol{H}_1(\mathcal{X}_1)u}$$

where $u(p_i) = v(p_i)\cdot\mathcal{X}_1$ for p_i appearing in α and $u(p_i)$ is an arbitrary element of $\boldsymbol{H}_1(\mathcal{X}_1)$ for p_i which do not appear in α.

The easy proof by induction with respect to the length of α is based on the following equalities:

(i) $\quad (X+Y)\cdot\mathcal{X}_1 = (X\cdot\mathcal{X}_1)+(Y\cdot\mathcal{X}_1)$,

(ii) $\quad (X\cdot Y)\cdot\mathcal{X}_1 = (X\cdot\mathcal{X}_1)\cdot(Y\cdot\mathcal{X}_1)$,

(iii) $\quad (X\to Y)\cdot\mathcal{X}_1 = X\cdot\mathcal{X}_1\to_1 Y\cdot\mathcal{X}_1$

(iv) $\quad (\neg X)\cdot\mathcal{X}_1 = \neg_1(X\cdot\mathcal{X}_1)$,

which hold for every $X, Y\in\boldsymbol{B}_1(\mathcal{X})$. Equalities (i), (ii) are obvious, (iii), (iv) follow immediately from (25).

Let $Z(i_1, \ldots, i_n, j_1, \ldots, j_n)$, where $j_r\neq j_s$ for $r\neq s$ and $j_r\neq i_s$ for $r, s = 1, \ldots, n$, be the set of formulas $\neg(p_{i_k}, p_{j_k})$, $k = 1, \ldots, n$.

We shall say that a formula β without the sign $\sim$ is provable in the Heyting propositional calculus from the set $Z(i_1, \ldots, i_n, j_1, \ldots, j_n)$ provided that it belongs to the least set $Cn_{\boldsymbol{H}}(Z(i_1, \ldots, i_n, j_1, \ldots, j_n))$ which contains the set $Z(i_1, \ldots, i_n, j_1, \ldots, j_n)$, all the formulas of the form (1)-(10) § 1 where α, β, γ are arbitrary formulas of $\mathcal{S}$ without the sign $\sim$, and which is closed under the operation of modus ponens.

The following lemmas are well known (cf. [4], [5], [10]):

4.5. *If β is a formula of $\mathcal{S}$ without the sign $\sim$ and $\beta \in Cn_{\boldsymbol{H}}(Z(i_1, \ldots, i_n, j_1, \ldots, j_n))$, then for every Heyting algebra of sets $\boldsymbol{H}(\mathcal{X}_1)$ and every valuation u in $\boldsymbol{H}(\mathcal{X}_1)$ such that $(\neg(p_{i_k}, p_{j_k}))_{\boldsymbol{H}(\mathcal{X}_1)u} = \mathcal{X}_1$ for $k = 1, \ldots, n$, the following condition is satisfied:*

$$\beta_{\boldsymbol{H}(\mathcal{X}_1)u} = \mathcal{X}_1.$$

4.6. *There exists a Heyting algebra of sets $\boldsymbol{H}(\mathcal{X}_1)$ and a valuation u in $\boldsymbol{H}(\mathcal{X}_1)$ such that*

$$(\neg(p_{i_k}, p_{j_k}))_{\boldsymbol{H}(\mathcal{X}_1)u} = \mathcal{X}_1 \quad \text{for} \quad k = 1, \ldots, n$$

and for every formula β of $\mathcal{S}$ without the sign $\sim$, $\beta \in Cn_{\boldsymbol{H}}(Z(i_1, \ldots, i_n, j_1, \ldots, j_n))$ if and only if $\beta_{\boldsymbol{H}(\mathcal{X}_1)u} = \mathcal{S}_1$.

The following theorem can easily be proved by using the method applied in the paper of Rasiowa [8].

4.7. *If $\alpha + \beta \in Cn_{\boldsymbol{H}}(Z(i_1, \ldots, i_n, j_1, \ldots, j_n))$, then either $\alpha \in Cn_{\boldsymbol{H}}(Z(i_1, \ldots, i_n, j_1, \ldots, j_n))$ or $\beta \in Cn_{\boldsymbol{H}}(Z(i_1, \ldots, i_n, j_1, \ldots, j_n))$.*

Let $\alpha(p_{i_1}, \ldots, p_{i_n})$ be an arbitrary formula of $\mathcal{S}$ and let $\alpha^0(p_{i_1}, \ldots, p_{i_n}, p_{j_1}, \ldots, p_{j_n})$ be the formula satisfying the condition of theorem 1.1 for the formula α.

4.8. *If $\boldsymbol{H}(\mathcal{X}_1)$ is a Heyting algebra of sets such that for the given formula $\alpha^0(p_{i_1}, \ldots, p_{i_n}, p_{j_1}, \ldots, p_{j_n})$ there exists a valuation u such that*

(i) $\alpha^0{}_{\boldsymbol{H}(\mathcal{X}_1)u} \neq \mathcal{X}_1$,

(ii) $(\neg(p_{i_k} \cdot p_{j_k}))_{\boldsymbol{H}(\mathcal{X}_1)u} = \mathcal{X}_1$ *for* $k = 1, \ldots, n$,

then for every $\mathcal{N}$-lattice of sets $\boldsymbol{B}(\mathcal{X})$ connected with $\boldsymbol{H}(\mathcal{X}_1)$ there exists a valuation v in $\boldsymbol{B}(\mathcal{X})$ such that

$$\alpha_{\boldsymbol{B}(\mathcal{X})v} \neq \mathcal{X}.$$

Let us suppose that $\boldsymbol{H}(\mathcal{X}_1)$ and u satisfy the hypothesis of the theorem for a formula $\alpha(p_{i_1}, \ldots, p_{i_n})$. Let $\boldsymbol{B}(\mathcal{X})$ be an $\mathcal{N}$-lattice of sets connected with $\boldsymbol{H}(\mathcal{X}_1)$ and let g be the involution of $\mathcal{X}$ determining $\boldsymbol{B}(\mathcal{X})$.

On account of (ii) we have

$$(p_{i_k} \cdot p_{j_k})_{\boldsymbol{H}(\mathcal{X}_1)u} = \Lambda.$$

Consequently the following subsets of $\mathcal{X}$

$$a_k = u(p_{i_k}) + (g(\mathcal{X}_1) - g(u(p_{j_k})) \qquad k = 1, \ldots, n$$

fulfil conditions (i), (ii), (iii) of § 3 and thus belong to $\boldsymbol{B}(\mathcal{X})$. Let v be the valuation in $\boldsymbol{B}(\mathcal{X})$ defined as follows:

$$v(p_{i_k}) = a_k, \quad v(p_{j_k}) = \sim a_k \text{ for } k = 1, \ldots, n \text{ and } v(p_i) = \mathcal{X}$$
$$\textit{for } i \neq i_k \text{ and } i \neq j_k.$$

Then by 4.4 and (i) we infer that

$$(\alpha^0(p_{i_1}, \ldots, p_{i_n}, p_{j_1}, \ldots, p_{j_n}))_{\boldsymbol{B}(\mathcal{X})v} \cdot \mathcal{X}_1$$
$$= (\alpha^0(p_{i_1}, \ldots, p_{i_n}, p_{j_1}, \ldots, p_{j_n}))_{\boldsymbol{H}(\mathcal{X}_1)u} \neq \mathcal{X}_1.$$

In consequence

$$(\alpha^0(p_{i_1}, \ldots, p_{i_n}, \sim p_{i_1}, \ldots, \sim p_{i_n}))_{\boldsymbol{B}(\mathcal{X})v}$$
$$= (\alpha^0(p_{i_1}, \ldots, p_{i_n}, p_{j_1}, \ldots, p_{j_n}))_{\boldsymbol{B}(\mathcal{X})v} \neq \mathcal{X}.$$

4.9. *A formula $\alpha(p_{i_1}, \ldots, p_{i_n})$ of $\mathcal{S}$ is provable in $\mathcal{S}$ if and only if the formula*

$$\alpha^0(p_{i_1}, \ldots, p_{i_n}, p_{j_1}, \ldots, p_{j_n}) \in Cn_{\boldsymbol{H}}(Z(i_1, \ldots, i_n, j_1, \ldots, j_n))$$

holds.

Sufficiency. Let us suppose that $\alpha(p_{i_1}, \ldots, p_{i_n}) \notin \mathcal{T}$. Consequently by 1.1 $\alpha^0(p_{i_1}, \ldots, p_{i_n}, p_{j_1}, \ldots, p_{j_n}) \notin \mathcal{T}$. By 4.2

(36) $$(\alpha^0(p_{i_1}, \ldots, p_{i_n}, \sim p_{i_1}, \ldots, \sim p_{i_n}))_{\boldsymbol{B}_0(\mathcal{X}_0)v_0} \neq \mathcal{X}_0$$

where $\boldsymbol{B}_0(\mathcal{X}_0)$ and v_0 are the $\mathcal{N}$-lattice of sets and the valuation in $\boldsymbol{B}_0(\mathcal{X}_0)$ appearing in the formulation of 4.2. Let $\boldsymbol{H}(\mathcal{X}_1)$ be the basic Heyting algebra of $\boldsymbol{B}_0(\mathcal{X}_0)$. Then by (4.4)

$$(\alpha^0(p_{i_1}, \ldots, p_{i_n}, \sim p_{i_1}, \ldots, \sim p_{i_n}))_{\boldsymbol{B}_0(\mathcal{X}_0)v_0} \cdot \mathcal{X}_1$$
$$= (\alpha^0(p_{i_1}, \ldots, p_{i_n}, p_{j_1}, \ldots, p_{j_n}))_{\boldsymbol{H}(\mathcal{X}_1)u}$$

where $u(p_{j_k}) = \sim v_0(p_{i_k}) \cdot \mathcal{X}_1$ for $k = 1, \ldots, n$ and $u(p_i) = v_0(p_i) \cdot \mathcal{X}_1$ for $i \neq j_k$ $k = 1, \ldots, n$. Consequently, by (36) and (iii), § 3, we obtain

(37) $$(\alpha_0(p_{i_1}, \ldots, p_{i_n}, p_{j_1}, \ldots, p_{j_n}))_{\boldsymbol{H}(\mathcal{X}_1)u} \neq \mathcal{X}_1.$$

On the other hand it is easy to verify by using (35) of paper [9] and (n_0) that

$$(X \cdot \sim X) \cdot \mathcal{X}_1 = \Lambda.$$

Hence $(\neg(p_{i_k}, p_{j_k}))_{\boldsymbol{H}(\mathcal{X}_1)u} = \neg_1 \Lambda = \mathcal{X}_1$. Thus, by (37) and 4.5 $\alpha^0(p_{i_1}, \ldots, p_{i_n}, p_{j_1}, \ldots, p_{j_n}) \notin Cn_{\boldsymbol{H}}(Z(i_1, \ldots, i_n, j_1, \ldots, j_n))$.

Necessity. Let us suppose, that $a_0(p_{i_1}, \ldots, p_{i_n}, p_{j_1}, \ldots, p_{j_n}) \notin Cn_{\boldsymbol{H}}(Z(i_1, \ldots, i_n, j_1, \ldots, j_n))$. Then by 4.6 there exist a Heyting algebra of sets $\boldsymbol{H}(\mathcal{X}_1)$ and a valuation u in this algebra such that $(\neg(p_{i_k}, p_{j_k}))_{\boldsymbol{H}(\mathcal{X}_1)u} = \mathcal{X}_1$ for $k = 1, \ldots, n$ and $a^0_{\boldsymbol{H}(\mathcal{X}_1)u} \neq \mathcal{X}_1$. Let $\boldsymbol{B}(\mathcal{X})$ be an $\mathcal{N}$-lattice of sets connected with $\boldsymbol{H}(\mathcal{X}_1)$. Then by 4.8 $a_{\boldsymbol{B}(\mathcal{X})} \neq \mathcal{X}$ for a suitable valuation v in $\boldsymbol{B}(\mathcal{X})$. Hence by 4.1 a is not provable in $\mathcal{S}$.

It follows from the definition of the set $Cn_{\boldsymbol{H}}(Z(i_1, \ldots, i_n, j_1, \ldots, j_n))$ and from the deduction theorem for the Heyting propositional calculus without the rule of substitution that, for a formula $a^0(p_{i_1}, \ldots, p_{i_n}, p_{j_1}, \ldots, p_{j_n})$, $j_r \neq j_s$ for $r \neq s$ and $j_r \neq i_s$, $r, s = 1, \ldots, n$, of $\mathcal{S}$ in which the sign $\sim$ does not appear, $a^0 \in Cn(Z(i_1, \ldots, i_n, j_1, \ldots, j_n))$ if and only if the formula $\neg(p_{i_1} \cdot p_{j_1}) \ldots \neg(p_{i_n} \cdot p_{j_n}) \to a^0$ is provable in the Heyting propositional calculus ([2]). Since the Heyting propositional calculus is decidable, we infer from 4.9 that the system $\mathcal{S}$ is also decidable (cf. [13]).

The following theorem can be proved by means of 4.7 and theorem 4.9:

4.10. *Let $a_1 + a_2$ be a formula provable in $\mathcal{S}$. Then either a_1 or a_2 is provable in $\mathcal{S}$.*

The first proof. Let us suppose that $p_{i_1}, \ldots, p_{i_n}$ are all propositional variables in $a_1 + a_2$. Let $(a_1+a_2)^0$ be constructed as in theorem 1.2, i. e. $(a_1+a_2)^0 = a_1^0 + a_2^0$. Since $a_1 + a_2 \in \mathcal{T}$, then by 4.9 $(a_1+a_2)^0 \in Cn_{\boldsymbol{H}}(Z(p_{i_1}, \ldots, p_{i_n}, p_{j_1}, \ldots, p_{j_n}))$. Consequently, by 4.7 at least one of the formulas a_1^0, a_2^0 belongs to this set. Let us suppose that $a_1^0 \in Cn_{\boldsymbol{H}}(Z(p_{i_1}, \ldots, p_{i_n}, p_{j_1}, \ldots, p_{j_n}))$ and that $p_{i_1}, \ldots, p_{i_m}$ $m \leqslant n$ are all propositional variables appearing in a_1. Then making use of 4.6 and 4.5 we easily infer that

$$a_1^0 \in Cn_{\boldsymbol{H}}(Z(i_1, \ldots, i_m, j_1, \ldots, j_m)).$$

In fact, let us suppose that a_1^0 does not belong to $Cn_{\boldsymbol{H}}(Z(i_1, \ldots, i_m, j_1, \ldots, j_m))$. By 4.6 there exists a Heyting algebra of sets $\boldsymbol{H}(\mathcal{X}_1)$ and a valuation u such that

$$(\neg(p_{i_k} \cdot p_{j_k}))_{\boldsymbol{H}(\mathcal{X}_1)u} = \mathcal{X}_1 \text{ for } k = 1, \ldots, m$$

and

$$(a_1^0(p_{i_1}, \ldots, p_{i_m}, p_{j_1}, \ldots, p_{j_m}))_{\boldsymbol{H}(\mathcal{X}_1)u} \neq \mathcal{X}_1.$$

Let us set

$$u_1(p_{i_k}) = u(p_{i_k}) \qquad k = 1, \ldots, m,$$
$$u_1(p_{j_k}) = u(p_{j_k}) \qquad k = 1, \ldots, m,$$
$$u_1(p_i) = \Lambda \quad \text{in the other cases.}$$

([2]) For the Heyting propositional calculus see [3].

Then it is easy to see that $(\neg(p_{i_k} \cdot p_{j_k}))_{\boldsymbol{H}(\mathcal{X}_1)u_1} = \mathcal{X}_1$ for $k = l, \ldots, n$ $a^0_{1\boldsymbol{H}(\mathcal{X}_1)u_1} \neq \mathcal{X}_1$. Thus by 4.5 $a_1^0 \notin Cn_{\boldsymbol{H}}(Z(i_1, \ldots, i_n, j_1, \ldots, j_n))$ which contradicts our hypothesis. Using 4.9 we infer that $a_1 \in \boldsymbol{T}$, which completes the proof.

The second proof. Let us suppose that $a_1 + a_2$ is provable in $\mathcal{S}$. Let $\boldsymbol{B}_1(\mathcal{X})$ be an $\mathcal{N}$-lattice of sets adequate for the system $\mathcal{S}$, and let v be a valuation in $\boldsymbol{B}_1(\mathcal{X})$ such that for every formula a of $\mathcal{S}$, a is provable if and only if $a_{\boldsymbol{B}_1(\mathcal{X})v} = \mathcal{X}$. Let us suppose that $\boldsymbol{B}_1(\mathcal{X})$ is a subalgebra of an $\mathcal{N}$-lattice of sets $\boldsymbol{B}(\mathcal{X})$ connected with a Heyting algebra $\boldsymbol{H}(\mathcal{X}_1)$. Consider the $\mathcal{N}$-lattice $\boldsymbol{B}_1(\mathcal{X}^0)$ constructed as in theorem 3.6. Let us set $h(X) = X \cdot \mathcal{X}$ for every $X \in \boldsymbol{B}_1(\mathcal{X}^0)$. Consider the following valuation w in $\boldsymbol{B}_1(\mathcal{X}_0)$ $w(p_i) = X_i \in \boldsymbol{B}_1(\mathcal{X}^0)$ where $h(X_i) = v(p_i)$, $i = 1, 2, \ldots$ It follows from 4.1 that

$$(a_1)_{\boldsymbol{B}_1(\mathcal{X}^0)w} + (a_2)_{\boldsymbol{B}_1(\mathcal{X}^0)w} = (a_1 + a_2)_{\boldsymbol{B}_1(\mathcal{X}^0)w} = \mathcal{X}^0.$$

By 3.6 (vii) at least one of the summands must be equal to $\mathcal{X}^0$. Let us suppose that $(a_1)_{\boldsymbol{B}_1(\mathcal{X}^0)w} = \mathcal{X}^0$. Since by 3.6 (vii) h is a homomorphism of $\boldsymbol{B}_1(\mathcal{X}^0)$ onto $\boldsymbol{B}_1(\mathcal{X})$ we infer that

$$h((a_1)_{\boldsymbol{B}_1(\mathcal{X}^0)w}) = (a_1)_{\boldsymbol{B}_1(\mathcal{X})(w)} = (a_1)_{\boldsymbol{B}_1(\mathcal{X})v} = \mathcal{X}^0 \cdot \mathcal{X} = \mathcal{X}$$

Consequently, a_1 is provable in $\mathcal{S}$.

We immediately get from this theorem and from (3), (7), (13) § 1 the following corollary:

4.11. *A formula $\sim(a \cdot \beta)$ is provable in $\mathcal{S}$ if and only if at least one of the formulas $\sim a$, $\sim \beta$ is provable in $\mathcal{S}$.*

We shall say that a Heyting algebra $\boldsymbol{H}(\mathcal{X}_1)$ fulfils the condition *Adq* provided that: for every formula $a(p_{i_1}, \ldots, p_{i_n}, p_{j_1}, \ldots, p_{j_n})$ where $i_r \neq i_s$, $j_r \neq j_s$ for $r \neq s$ and $i_r \neq j_s$, $r, s = 1, \ldots, n$, in which the sign $\sim$ does not appear, if $a \notin Cn_{\boldsymbol{H}}(Z(i_1, \ldots, i_n, j_1, \ldots, j_n))$ then there exists a valuation u in $\boldsymbol{H}(\mathcal{X}_1)$ such that

$$(38) \qquad (\neg(p_{i_k} \cdot p_{j_k}))_{\boldsymbol{H}(\mathcal{X}_1)u} = \mathcal{X}_1 \quad \text{for} \quad k = 1, \ldots, n$$

and

$$(39) \qquad a_{\boldsymbol{H}(\mathcal{X}_1)u} \neq \mathcal{X}_1.$$

4.12. *Every $\mathcal{N}$-lattice of sets $\boldsymbol{B}(\mathcal{X})$ connected with a Heyting algebra $\boldsymbol{H}(\mathcal{X}_1)$ satisfying the condition Adq is adequate for the system $\mathcal{S}$. More precisely, a formula a of $\mathcal{S}$ is provable in $\mathcal{S}$ if and only if $a_{\boldsymbol{B}(\mathcal{X})v} = \mathcal{X}$ for every valuation v in $\boldsymbol{B}(\mathcal{X})$.*

The proof immediately follows from 4.1, 4.9 and 4.8.

4.13. *The Heyting algebra $\boldsymbol{H}(E)$ of all open subsets of the real line fulfils the condition Adq.*

Let us suppose that $\alpha(p_{i_1}, \ldots, p_{i_n}, p_{j_1}, \ldots, p_{j_n}) \notin Cn_{\boldsymbol{H}}(Z(i_1, \ldots, i_n, j_1, \ldots, j_n))$. Then the formula $\beta = \neg(p_{i_1} \cdot p_{j_1}) \ldots \neg(p_{i_n} \cdot p_{j_n}) \to \alpha$ is not provable in the Heyting propositional calculus. Since $\boldsymbol{H}(E)$ is adequate for this propositional calculus (³), there exists a valuation u in $\boldsymbol{H}(E)$ such that $\beta_{\boldsymbol{H}(E)u} \neq E$. Consequently

$$(\neg(p_{i_1} \cdot p_{j_1}) \ldots \neg(p_{i_n} \cdot p_{j_n}))_{\boldsymbol{H}(E)u} \not\subset \alpha_{\boldsymbol{H}(E)u}.$$

Thus there exists a point $x \in E$ such that

$$x \in (\neg(p_{i_1} \cdot p_{j_1}) \ldots \neg(p_{i_n} \cdot p_{j_n}))_{\boldsymbol{H}(E)u} - \alpha_{\boldsymbol{H}(E)u}.$$

Hence, there exists an open interval $\mathcal{X}_1 \subset E$, such that $x \in \mathcal{X}_1$ and

$$\mathcal{X}_1 \cdot \sum_{k=1}^{n} (u(p_{i_k}) \cdot u(p_{j_k})) = \Lambda.$$

Let us put $w(p_i) = u(p_i) \cdot \mathcal{X}_1$ and let $\boldsymbol{H}(\mathcal{X}_1)$ be the Heyting algebra of all open subsets of $\mathcal{X}_1$. Then

$$\mathcal{X}_1 \cdot \alpha_{\boldsymbol{H}(E)u} = \alpha_{\boldsymbol{H}(\mathcal{X}_1)w} \neq \mathcal{X}_1$$

since the mapping $h(G) = \mathcal{X}_1 \cdot G$ for every $G \in \boldsymbol{H}(E)$ is a homomorphism of $\boldsymbol{H}(E)$ onto $\boldsymbol{H}(\mathcal{X}_1)$ and $x \in \mathcal{X}_1 - \alpha_{\boldsymbol{H}(E)u}$. On the other hand

$$(\neg(p_{i_k} \cdot p_{j_k}))_{\boldsymbol{H}(\mathcal{X}_1)w} = \mathrm{Int}(\mathcal{X}_1 - w(p_{i_k}) \cdot w(p_{j_k})) = \mathrm{Int}(\mathcal{X}_1) = \mathcal{X}_1.$$

The Heyting algebras $\boldsymbol{H}(E)$ and $\boldsymbol{H}(\mathcal{X}_1)$ being isomorphic, there exists a valuation u_0 in $\boldsymbol{H}(E)$, such that

$$(\neg(p_{i_k} \cdot p_{j_k}))_{\boldsymbol{H}(E)u_0} = E$$

and

$$\alpha_{\boldsymbol{H}(E)u_0} \neq E.$$

An analogical proof holds also for the Heyting algebra of all open subsets of the n-dimensional Euclidean spaces and for the Heyting algebra of all open subsets of the Cantor discontinuum. It follows immediately from 4.12 and 4.13 that

4.14. *Every $\mathcal{N}$-lattice of sets $\boldsymbol{B}(\mathcal{X})$ connected with the Heyting algebra of all open subsets of the n-dimensional Euclidean space, or of the Cantor discontinuum is adequate for the system $\mathcal{S}$* (⁴).

(³) Cf. e. g. [4]. This result is due to Tarski (Fundamenta Mathematicae 31 (1938), p. 103-134).

(⁴) The proof is based on the results of McKinsey-Tarski [4] stating that the Heyting algebra of all open subsets of the n-dimensional Euclidean space or of the Cantor discontinuum is functionally free.

The following known theorem (cf. [13]) can easily be deduced from the algebraic characterization of the system $\mathcal{S}$.

4.15. *A formula α of $\mathcal{S}$ without the sign $\sim$ is provable in $\mathcal{S}$ if and only if it is provable in the Heyting propositional calculus.*

The sufficiency of this condition is obvious. To prove the necessity let us suppose that α is not provable in the Heyting propositional calculus. Then there exist a Heyting algebra of sets $\boldsymbol{H}(\mathcal{X}_1)$ and a valuation u in $\boldsymbol{H}(\mathcal{X}_1)$ such that

$$\alpha_{\boldsymbol{H}(\mathcal{X}_1)u} \neq \mathcal{X}_1$$

Let $\boldsymbol{B}(\mathcal{X})$ be an $\mathcal{N}$-lattice of sets connected with $\boldsymbol{H}(\mathcal{X}_1)$ such that $\mathcal{X}_1 \cdot g(\mathcal{X}_1) = \Lambda$, where g is the involution determining $\boldsymbol{B}(\mathcal{X})$. It is easy to verify that, if $X \in \boldsymbol{H}(\mathcal{X}_1)$, then $X + g(\mathcal{X}_1) \in \boldsymbol{B}(\mathcal{X})$. In fact $X + g(\mathcal{X}_1)$ fulfils conditions (i), (ii), (iii) § 3. Moreover, the valuation v in $\boldsymbol{B}(\mathcal{X})$ defined as

$$v(p_i) = u(p_i) + g(\mathcal{X}_1)$$

has the following property:

$\beta_{\boldsymbol{B}(\mathcal{X}),v} = \beta_{\boldsymbol{H}(\mathcal{X}_1)u} + g(\mathcal{X}_1)$ for every formula β of $\mathcal{S}$ without the sign $\sim$. Consequently

$$\alpha_{\boldsymbol{B}(\mathcal{X})v} \neq \mathcal{X} \cdot \mathcal{X}_1 + g(\mathcal{X}_1).$$

Thus by 4.1 α is not provable in $\mathcal{S}$.

§ 5. The $\mathcal{N}$-lattice of sets $\boldsymbol{B}_1(\mathcal{Y})$ isomorphic with the Lindenbaum algebra of the system $\mathcal{S}^*$. Given arbitrary formulas α, β of $\mathcal{S}^*$ we shall write $\alpha \simeq \beta$ provided that the formula $\alpha \equiv \beta \in \mathcal{T}^*$. It is easy to prove that the relation $\simeq$ is a congruence relation in the sense of modern algebra. For every formula α let $|\alpha|$ denote the class of all formulas β of $\mathcal{S}^*$ such that $\alpha \simeq \beta$. Let A_0 be the set of all cosets $|\alpha|$ where α is a formula of $\mathcal{S}^*$. We define in A_0 the algebraical operations $+, \cdot, \sim, \to, \neg$ as $|\alpha| \circ |\beta| = |\alpha \circ \beta|$ if $\circ$ is one of the binary logical operations of $\mathcal{S}^*$ and $\circ|\alpha| = |\circ\alpha|$ if $\circ$ is one of the unary logical operations of $\mathcal{S}^*$. If $\alpha \in \mathcal{T}^*$ and $\beta \in \mathcal{T}^*$ then $\alpha \simeq \beta$. The element $|\alpha|$ where $\alpha \in \mathcal{T}^*$ will be denoted by e_0. It is known (cf. [9]) that

5.1. *The algebra $\boldsymbol{L}^* = \langle A_0, e_0, +, \cdot, \sim, \to, \neg \rangle$ is an $\mathcal{N}$-lattice. The inclusion $|\alpha| \subset |\beta|$ holds if and only if $\alpha \to \beta \in \mathcal{T}^*$ and $\sim\beta \to \sim\alpha \in \mathcal{T}^*$. The relation $|\alpha| < |\beta|$ holds if and only if $\alpha \to \beta \in \mathcal{T}^*$.*

Let $\boldsymbol{B}_1(\mathcal{Y})$ be an $\mathcal{N}$-lattice of sets isomorphic with $\boldsymbol{L}^*$ and let h^* be the isomorphism of $\boldsymbol{L}^*$ onto $\boldsymbol{B}_1(\mathcal{Y})$. We shall assume that $\boldsymbol{B}_1(\mathcal{Y})$ is a subalgebra of an $\mathcal{N}$-lattice of sets $\boldsymbol{B}(\mathcal{Y})$ connected with $\boldsymbol{H}(\mathcal{Y}_1)$ and that $\boldsymbol{H}_1(\mathcal{Y}_1)$ is the basic Heyting subalgebra of $\boldsymbol{B}_1(\mathcal{Y})$.

In the sequel the following notation we shall use. Given a lattice A and a set $\{a_\mu\}_{\mu\epsilon M}$ of its elements we shall denote by $(A)\sum_{\mu\epsilon M} a_\mu$ and by $(A)\prod_{\mu\epsilon M} a_\mu$ the sum of those elements in A and their product in A, respectively.

5.2. *For every formula α of $\mathcal{S}^*$ the following equalities hold:*

(i) $$(\boldsymbol{B}_1(\mathcal{Y}))\sum_{p\epsilon I_0} h^*\left(\left|\alpha\binom{x_p}{x_k}\right|\right) = h^*\left(\left|\sum_{x_k}\alpha\right|\right),$$

(ii) $$(\boldsymbol{B}_1(\mathcal{Y}))\prod_{p\epsilon I_0} h^*\left(\left|\alpha\binom{x_p}{x_k}\right|\right) = h^*\left(\left|\prod_{x_k}\alpha\right|\right),$$

(iii) $$(\boldsymbol{H}_1(\mathcal{Y}_1))\sum_{p\epsilon I_0} \mathcal{Y}_1\cdot h^*\left(\left|\alpha\binom{x_p}{x_k}\right|\right) = \mathcal{Y}_1\cdot h^*\left(\left|\sum_{x_k}\alpha\right|\right),$$

(iv) $$(\boldsymbol{H}_1(\mathcal{Y}_1))\prod_{p\epsilon I_0} \mathcal{Y}_1\cdot h^*\left(\left|\alpha\binom{x_p}{x_k}\right|\right) = \mathcal{Y}_1\cdot h^*\left(\left|\prod_{x_k}\alpha\right|\right),$$

where $\alpha\binom{x_p}{x_k}$ is a formula obtained from α by the substitution of x_p for x_k, all necessary changes of bound variables being performed in α before the operation of substitution has been applied.

In fact, to show (i) let us notice that by 17*, 18*, and 5.1

(40) $$h^*\left(\left|\alpha\binom{x_p}{x_k}\right|\right) < h^*\left(\left|\sum_{x_k}\alpha\right|\right) \quad \text{for} \quad p\epsilon I_0,$$

(41) $$\sim h^*\left(\left|\sum_{x_k}\alpha\right|\right) < \sim h^*\left(\left|\alpha\binom{x_p}{x_k}\right|\right) \quad \text{for} \quad p\epsilon I_0.$$

Consequently,

(42) $$h^*\left(\left|\alpha\binom{x_p}{x_k}\right|\right) \subset h^*\left(\left|\sum_{x_k}\alpha\right|\right) \quad \text{for} \quad p\epsilon I_0.$$

On the other hand let us suppose that for some β

$$h^*\left(\left|\alpha\binom{x_p}{x_k}\right|\right) \subset h^*(|\beta|) \quad \text{for every} \quad p\epsilon I_0.$$

Hence, by 5.1 we infer that for some p such that x_p is not free in β.

$$\alpha\binom{x_p}{x_k}\to\beta\,\epsilon\,\mathcal{T}^* \quad \text{and} \quad \sim\beta\to\sim\alpha\binom{x_p}{x_k}\epsilon\,\mathcal{T}^*.$$

Using the rule (v), § 2, we obtain by means of 5.1

$$h^*\left(\left|\sum_{x_k}\alpha\right|\right) \subset h^*(|\beta|)$$

which, with (42), prove (i).

In a similar way it can be proved that (ii) holds.

To show (iii) let us notice that by (40) and (26)

(43) $$\mathcal{Y}_1\cdot h^*\left(\left|\alpha\binom{x_p}{x_k}\right|\right) \subset \mathcal{Y}_1\cdot h^*\left(\left|\sum_{x_k}\alpha\right|\right) \quad \text{for every} \quad p\epsilon I_0.$$

Now let us suppose that for some β

$$\mathcal{Y}_1\cdot h^*\left(\left|\alpha\binom{x_p}{x_k}\right|\right) \subset \mathcal{Y}_1\cdot h^*(|\beta|), \quad \text{for every} \quad p\epsilon I_0.$$

Consequently by (26)

$$h^*\left(\left|\alpha\binom{x_p}{x_k}\right|\right) < h^*(|\beta|).$$

Thus for some p such that x_p is not free in β

$$\alpha\binom{x_p}{x_k}\to\beta\,\epsilon\,\mathcal{T}^*.$$

Using the rule (iv) § 2 and 5.1 we infer that

$$h^*\left(\left|\sum_{x_k}\alpha\right|\right) < h^*(|\beta|).$$

Consequently, by (26)

$$\mathcal{Y}_1\cdot h^*\left(\left|\sum_{x_k}\alpha\right|\right) \subset \mathcal{Y}_1\cdot h^*(|\beta|).$$

Hence on account of (43) condition (iii) holds.

The proof of (iv) is analogous.

§ 6. Models of the system $\mathcal{S}^*$. Let $\boldsymbol{B}_1(\mathcal{X})$ be an $\mathcal{N}$-lattice of sets and let J be a non empty set. Every mapping $(J, \boldsymbol{B}_1(\mathcal{X}))\mathfrak{M}$ (or briefly $\mathfrak{M}$ of all functional variables F_k, $k\epsilon I_0$ into the sets $(J, \boldsymbol{B}_1(\mathcal{X}))\boldsymbol{E}_{\nu(k)}$ of all $\nu(k)$-argument functions defined on J with values in $\boldsymbol{B}_1(\mathcal{X})$ is said to be a *realization* of $\mathcal{S}^*$ in the set J and algebra $\boldsymbol{B}_1(\mathcal{X})$. Let v be a valuation of all individual variables of $\mathcal{S}^*$ in J. Every formula α of $\mathcal{S}^*$ may be interpreted as a functional $\alpha_{(J\boldsymbol{B}(\mathcal{X}))\mathfrak{M}}$ or briefly $\alpha_{\mathfrak{M}}$ determined on J with values in $\boldsymbol{B}(\mathcal{X})$ by treating

(a) all individual variables x_k, $k\epsilon I_0$ as variables running over J,

(b) every functional variable F_k as $\mathfrak{M}(F_k)$,

(c) each of the logical connectives $+$, $\cdot$, $\sim$, $\to$, $\neg$ as the corresponding operation in $\boldsymbol{B}_1(\mathcal{X})$,

(d) the quantifiers $\sum_{x_k}$, $\prod_{x_k}$ as the signs of infinite sums $(\boldsymbol{B}_1(\mathcal{X}))\sum_{x_k\epsilon J}$ and of infinite products $(\boldsymbol{B}_1(\mathcal{X}))\prod_{x_k\epsilon J}$, respectively.

A realization $(J, \boldsymbol{B}_1(\mathcal{X}))\mathfrak{M}$ (or briefly $\mathfrak{M}$, if the set J and the algebra $\boldsymbol{B}_1(\mathcal{X})$ are fixed) will be called a *pseudomodel of* $\mathcal{S}^*$ if the following condition is satisfied:

(m_1) for every formula α of $\mathcal{S}^*$ and for every valuation v in J there exist all infinite sums and products appearing in $\alpha_{\mathfrak{M}}$ which correspond to the logical quantifiers by the values of arguments of $\alpha_{\mathfrak{M}}$ fixed by the valuation v.

If $\mathfrak{M}$ is a pseudomodel of $\mathcal{S}^*$ in a set J and in an $\mathcal{N}$-lattice $\boldsymbol{B}_1(\mathcal{X})$, then for every formula α and for every valuation v in J the symbol $\alpha_{\mathfrak{M}v}$ will denote the value of $\alpha_{\mathfrak{M}}$ by the values of its arguments fixed by the valuation v.

A pseudomodel $\mathfrak{M}$ of $\mathcal{S}^*$ in a set J and in an $\mathcal{N}$-lattice of sets $\boldsymbol{B}_1(\mathcal{X})$ having $\boldsymbol{H}_1(\mathcal{X}_1)$ as its basic Heyting algebra, is said to be a *model* of $\mathcal{S}^*$ provided that the following conditions are fulfilled:

(m_2) if
$$\Big(\sum_{x_k}\alpha(x_k)\Big)_{\mathfrak{M}v} = (\boldsymbol{B}_1(\mathcal{X}))\sum_{v(x_k)\in J}\alpha_{\mathfrak{M}v} = X, \text{ then}$$
$$\mathcal{X}_1\cdot X = (\boldsymbol{H}(\mathcal{X}_1))\sum_{v(x_k)\in J}(\alpha_{\mathfrak{M}v}\cdot\mathcal{X}_1);$$

(m_3) if
$$\Big(\prod_{x_k}\alpha(x_k)\Big)_{\mathfrak{M}v} = (\boldsymbol{B}_1(\mathcal{X}))\prod_{v(x_k)\in J}\alpha_{\mathfrak{M}v} = X, \text{ then}$$
$$\mathcal{X}_1\cdot X = (\boldsymbol{H}(\mathcal{X}_1))\prod_{v(x_k)\in J}(\alpha_{\mathfrak{M}v}\cdot\mathcal{X}_1).$$

6.1. *If* $\alpha\in\mathcal{T}^*$, *then* $\alpha_{\mathfrak{M}v} = \mathcal{X}$ *for every model* $(J, \boldsymbol{B}_1(\mathcal{X}))\mathfrak{M}$ *of* $\mathcal{S}^*$ *in every set* J *and in every* $\mathcal{N}$*-lattice of sets* $\boldsymbol{B}_1(\mathcal{X})$, *and for every valuation* v *in* J.

The easy proof by induction with respect to the length of a proof of α making use of 4.1, (n_1), (n_2), (iii) § 3, (m_1), (m_2), (m_3) and (3.7) is omitted.

Let us now use the notation of § 5 and let $\mathfrak{N}$ be the mapping associating with every F_k, $k\in I_0$ the $\nu(k)$-argument function φ_k determined on I_0 with values in $\boldsymbol{B}_1(\mathcal{Y})$ which will be defined as follows: $\varphi_k(i_1, \ldots, i_{\nu(k)}) = h^*(|F_k(x_{i_1}, \ldots, x_{i_{\nu(k)}})|)\in\boldsymbol{B}_1(\mathcal{Y})$. The following theorem may easily be proved by the induction with respect to the length of a formula α with the aid of theorem 5.2

6.2. $\mathfrak{N}$ *is a model of* $\mathcal{S}^*$ *in the set* I_0 *of positive integers and in the* $\mathcal{N}$*-lattice of sets* $\boldsymbol{B}_1(\mathcal{Y})$ *isomorphic with the Lindenbaum algebra* $\boldsymbol{L}^*$ *of the system* $\mathcal{S}^*$.

More exactly, *for every formula* α *and every valuation* v
$$\alpha_{\mathfrak{N}v} = h^*(|\alpha_v|),$$

where α_v *is a formula obtained from* α *by substituting the individual variable* $x_{v(x_k)}$ *for* x_k (the necessary changes of bound variables being performed before the operation of substitution).

In particular, for the valuation $v_0(x_k) = k$, $k\in I_0$, *we obtain*

$$\alpha_{\mathfrak{N}v_0} = h^*(|\alpha|). \tag{44}$$

The following theorem results immediately from 6.2 and 6.1.

6.3. *A formula* α *of* $\mathcal{S}^*$ *is provable if and only if*
$$\alpha_{\mathfrak{N}v_0} = \mathcal{Y}.$$

It is easy to show

6.4. *Let* $\beta = \alpha\binom{x_p}{x_k}$. *Then for every model* $\mathfrak{M}$ *and valuation* v
$$\beta_{\mathfrak{M}v} = \alpha_{\mathfrak{M}v'}$$
where $v'(x_i) = v(x_i)$ *for* $i\neq k$ *and* $v'(x_k) = v(x_p)$.

Let $\mathfrak{M}$ be a model of $\mathcal{S}^*$ in a set $J\neq\Lambda$ and in an $\mathcal{N}$-lattice of sets $\boldsymbol{B}_1(\mathcal{X})$ and let $\boldsymbol{B}_1(\mathcal{X}^0)$ be an $\mathcal{N}$-lattice of sets appearing in the formulation of theorem 3.6 (v). In the sequel we shall use the notation introduced to describe the construction of $\boldsymbol{B}_1(\mathcal{X}^0)$. Let us suppose that for every $k\in I_0$
$$\mathfrak{M}(F_k) = \psi_k.$$

We shall define a realization $\mathfrak{M}^0$ of $\mathcal{S}^*$ in the set J and in $\boldsymbol{B}_1(\mathcal{X}^0)$ as follows:

$\mathfrak{M}^0(F_k) = \psi_k^0$ where
$$\psi_k^0(j_1, \ldots, j_{\nu(k)}) = \begin{cases}\mathcal{X}^0 & \text{if } \psi_k(j_1, \ldots, j_{\nu(k)}) = \mathcal{X},\\ \psi_k(j_1, \ldots, j_{\nu(k)}) + \{x_2\} & \text{in the other case.}\end{cases}$$

6.5. $\mathfrak{M}^0$ *is a model of* $\mathcal{S}^*$. *More precisely, for every formula* α *of* $\mathcal{S}^*$ *and for every valuation* v *in* J *one of the following conditions holds:*
$$\alpha_{\mathfrak{M}^0v} = \Lambda, \quad \alpha_{\mathfrak{M}^0v} = \mathcal{X}^0, \quad \alpha_{\mathfrak{M}^0v} = \alpha_{\mathfrak{M}v} + \{x_2\}.$$
Moreover, if $\alpha_{\mathfrak{M}^0v} = \Lambda$ then $\alpha_{\mathfrak{M}v} = \Lambda$ *and if* $\alpha_{\mathfrak{M}^0v} = \mathcal{X}^0$, *then* $\alpha_{\mathfrak{M}v} = \mathcal{X}$.

The proof by induction with respect to the length of α with the aid of (29)-(35) of theorem 3.6 and the definition of $\boldsymbol{B}_1(\mathcal{X}^0)$ is omitted.

Let $h(X) = X\cdot\mathcal{X}$ for $X\in\boldsymbol{B}_1(\mathcal{X}^0)$. Then by 3.6 h is a homomorphism of $\boldsymbol{B}_1(\mathcal{X}^0)$ onto $\boldsymbol{B}_1(\mathcal{X})$. It is easy to show that

6.6. *For every set* $\{X_\mu\}_{\mu\in M}$ *of elements of* $\boldsymbol{B}_1(\mathcal{X}^0)$ *the existence of the sum* $(\boldsymbol{B}_1(\mathcal{X}^0))\sum_{\mu\in M}X_\mu$ *(the product* $(\boldsymbol{B}_1(\mathcal{X}^0))\prod_{\mu\in M}X_\mu$*) is equivalent to the existence of the sum* $(\boldsymbol{B}_1(\mathcal{X}))\sum_{\mu\in M}h(X_\mu)$ *(the product* $(\boldsymbol{B}_1(\mathcal{X}))\prod_{\mu\in M}h(X_\mu)$*) and the following* *equalities hold*:

$$h\Big(\big(\boldsymbol{B}_1(\mathcal{X}^0)\big)\sum_{\mu\epsilon M} X_\mu\Big) = \big(\boldsymbol{B}_1(\mathcal{X})\big)\sum_{\mu\epsilon M} h(X_\mu),$$

$$h\Big(\big(\boldsymbol{B}_1(\mathcal{X}^0)\big)\prod_{\mu\epsilon M} X_\mu\Big) = \big(\boldsymbol{B}_1(\mathcal{X})\big)\prod_{\mu\epsilon M} h(X_\mu).$$

The following theorem may easily be deduced from 6.5 and 6.6

6.7. *For every formula β of $\mathcal{S}^*$ and every valuation v in J^0, $h(\beta_{\mathfrak{M}^0,v}) = \beta_{\mathfrak{M},v}$.*

We shall use the previous theorems to show the following property of formulas of $\mathcal{S}^*$:

6.8. (i) *If a formula $\alpha+\beta$ is provable in $\mathcal{S}^*$, then either α or β is provable in $\mathcal{S}^*$.*

(ii) *If a formula $\sum\limits_{x_k}\alpha(x_k)$ is provable in $\mathcal{S}^*$, then there exists a positive integer p such that the formula $\alpha\binom{x_p}{x_k}$ is provable in $\mathcal{S}^*$.*

We shall show only (ii), since the proof of (i) is quite analogous to that of (ii). Let us suppose that $\sum\limits_{x_k}\alpha(x_k)\epsilon\mathcal{T}^*$. Consider the model $\mathfrak{N}$ in the set I_0 and in the $\mathcal{N}$-lattice of sets $\boldsymbol{B}_1(\mathcal{Y})$ isomorphic with the Lindenbaum algebra $\boldsymbol{L}^*$ of the system $\mathcal{S}^*$, which appears in the formulation of 6.2. Let us set $v(x_k) = k$ for every $k = 1, 2, \ldots$ Let $\mathfrak{N}^0$ be the model of $\mathcal{S}^*$ in I_0 and $\boldsymbol{B}_1(\mathcal{Y}^0)$ constructed for $\mathfrak{N}$ in the same way as a model $\mathfrak{M}^0$ has been constructed for a model $\mathfrak{M}$ to formulate theorem 6.5.

The $\mathcal{N}$-lattice of sets $\boldsymbol{B}_1(\mathcal{Y}^0)$ is constructed for $\boldsymbol{B}_1(\mathcal{Y})$ just as $\boldsymbol{B}_1(\mathcal{X}^0)$ has been constructed for $\boldsymbol{B}_1(\mathcal{X})$ (cf. 3.6). It follows from 6.1 that $(\sum\limits_{x}\alpha)_{\mathfrak{N}^0v} = \mathcal{Y}^0$, i. e.

$$\big(\boldsymbol{B}_1(\mathcal{Y}^0)\big)\sum_{v(x_k)\epsilon I^0}\alpha_{\mathfrak{N}^0v} = \mathcal{Y}^0. \tag{45}$$

Consequently, by 3.6 (vii) there exists $p\,\epsilon I_0$ such that

$$\alpha_{\mathfrak{N}^0v'} = \mathcal{Y}^0 \quad \text{where} \quad v'(x_i) = \begin{cases} v(x_i) & \text{for } i \neq k, \\ p & \text{for } i = k. \end{cases}$$

Thus, by 6.4, $\Big(\alpha\binom{x_p}{x_k}\Big)_{\mathfrak{N}^0v} = \mathcal{Y}^0$.

Hence, using 6.7 we obtain

$$\mathcal{Y} = h(\mathcal{Y}^0) = \Big(\alpha\binom{x_p}{x_k}\Big)_{\mathfrak{N}v}.$$

In consequence, by 6.3, $\alpha\binom{x_p}{x_k}\epsilon\mathcal{T}^*$.

Since the system $\mathcal{S}$ of the constructive propositional calculus with strong negation is decidable, it is easy to show by a method similar to that used in the paper of Rasiowa and Sikorski [11] that

6.9. *Each formula β from $\mathcal{S}^*$ of the form*

$$\beta = \overset{(1)}{\underset{x_{k_1}}{\Xi}}\ldots\overset{(n)}{\underset{x_{k_n}}{\Xi}}\alpha$$

where α contains no quantifiers and $\overset{(i)}{\underset{x_{k_i}}{\Xi}}$ is either the sign of $\sum\limits_{x_{k_i}}$ or $\prod\limits_{x_{k_i}}$ is decidable.

It follows immediately from the last theorem and from (3), (7), (13) § 1, (21*), (22*) § 2 that

6.10. *A formula $\sim(\alpha\cdot\beta)$ is provable in $\mathcal{S}^*$ if and only if at least one of the formulas $\sim\alpha$, $\sim\beta$ is provable in $\mathcal{S}^*$. A formula $\sim\prod\limits_{x_k}\alpha(x_k)$ is provable in $\mathcal{S}^*$ if and only if there exists x_p such that a formula $\sim\alpha\binom{x_p}{x_k}$ is provable in $\mathcal{S}^*$.*

We shall now prove that

6.11. *A formula α of $\mathcal{S}^*$ without the sign $\sim$ is provable in $\mathcal{S}^*$ if and only if it is provable in the Heyting functional calculus.*

The sufficiency of this condition is obvious. To prove the necessity let us suppose that α is not provable in the Heyting functional calculus ([5]). Then it is known (see [10]) that there exists a complete Heyting algebra of sets $\boldsymbol{H}(\mathcal{X}_1)$ and a realization $\mathfrak{M}$ of all functional variables of $\mathcal{S}^*$ in the set I_0 and $\boldsymbol{H}(\mathcal{X}_1)$ such that

$$\alpha_{\boldsymbol{H}(\mathcal{X}_1)v} \neq \mathcal{X}_1, \quad \text{where} \quad v(x_k) = k \quad \text{for} \quad k\,\epsilon\, I_0.$$

Let $\boldsymbol{B}(\mathcal{X})$ be an $\mathcal{N}$-lattice of sets connected with $\boldsymbol{H}(\mathcal{X}_1)$ such that $\mathcal{X}_1\cdot g(\mathcal{X}_1) = \Lambda$, where g is the involution determining $\boldsymbol{B}(\mathcal{X})$. Let us suppose that $\mathfrak{M}(F_k) = \varphi_k$, $k\,\epsilon\, I_0$. Let us set for every $k\,\epsilon\, I_0$

$$\mathfrak{M}^*(F_k) = \varphi_k^*, \quad \text{where} \quad \varphi_k^*(i_1, \ldots, i_{\nu(k)}) = \varphi_k(i_1, \ldots, i_{\nu(k)}) + g(\mathcal{X}_1).$$

It is easy to verify that $\mathfrak{M}^*$ is a model of $\mathcal{S}^*$ and for every formula β without the sign $\sim$

$$\beta_{\mathfrak{M}^*v} = \beta_{\mathfrak{M}v} + g(\mathcal{X}_1).$$

Consequently

$$\alpha_{\mathfrak{M}^*v} \neq \mathcal{X} = \mathcal{X}_1 + g(\mathcal{X}_1).$$

Thus, by 6.1. α is not provable in $\mathcal{S}^*$.

([5]) For the Heyting functional calculus see [3].

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ON CONVERGENCE OF MAPPINGS

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1. Introduction. The idea of a mapping, that is a single valued continuous transformation, as an extension of the function concept lies deep in the history of topology. Indeed it is closely interlaced with the very beginnings of topology, as is apparent to any student of complex function theory with its strong emphasis on mappings generated by differentiable functions. In more recent times, however, one of the most powerful and stimulating influences in the development of topology and its applications has been the method of generation of mappings by decomposition of the domain space into disjoint closed sets, together with the dual operation of generating a decomposition of a domain space by means of a given mapping defined on that space. The early recognition by Kuratowski [1] of the equivalence of these operations in an appropriate setting and his formulation of some of the then current work on upper semi-continuous decompositions in terms of mappings surely represents a distinct landmark in the development of Analytic Topology and has lead to major advances in this area of mathematical work. It is a privilege and a pleasure, therefore, for the author to dedicate this paper to his long-time friend and colleague Casimir Kuratowski on the occasion of the 40th anniversary of his first mathematical publication. The author's mathematical life and work have been immeasurably stimulated and enriched through personal and professional association with this great mathematician and by his masterful and exceptional skill in topological writing and exposition.

We shall be concerned in this paper with sequences of mappings from one locally compact separable metric space to another*. Conditions for the almost uniform convergence of such sequences having some applicability in the case of function sequences will be studied. The existence

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