can always choose x' so that $\int_0^{x'} p(x)dx > 0$, we can conclude that for sufficiently large a the considered integral becomes positive. The lemma is thus proved.

Proof of theorem 4. Let us consider two domains D_i and D_j. If $p(x)$ is the probability density of the distance of two points chosen at random and independently of D_i and D_j, respectively, and if x_0 is the distance between the centres of gravity of D_i and D_j, then we can apply the lemma to $p(x)$ and x_0. Thus for sufficiently large a we get inequality (6.5). This, however, is nothing else but another form of inequality (5.3), for a given i and j. Since this reasoning is applicable to each pair of domains D_i, D_j, we can choose a so large that inequalities (5.3) hold for all i and j, $i \neq j$. This proves our theorem.

Finally let us note that the following question is still open:

P 254. Is always systematic sampling, for isotropic processes $y(p)$ with the exponential correlation function and domains D composed of squares or of regular hexagons, more efficient than stratified sampling?

REFERENCES

[1] W. G. Cochran, *Relative accuracy of systematic and stratified samples for a certain class of populations*, Annals of Mathematical Statistics 17 (1946), p. 164-177.

[2] D. J. Finney, *Random and systematic sampling in timber surveys*, Forestry 22 (1948), p. 64-99; 23 (1950), p. 96-111 (quoted after [9]).

[3] J. Hájek, *Příspěvky k teorii statistického odhadu*, kandidátská disertace, Praha 1955. *(Some contributions to the theory of estimation,* Thesis*).*

[4] В. М. Крейтер, *Поиски и разведки полезных ископаемых*, Москва 1940.

[5] B. Matern, *Metoder att uppskata noggranhetten vid linje- och provytetaxering (Methods of estimating the accuracy of line and sample plot surveys)*, Meddelanden från Statens Skogsforskningsinstitut 36, 1 (1947), p. 1-138.

[6] M. H. Quenouille, *Problems in plane sampling*, Annals of Mathematical Statistics 20 (1949), p. 355-375.

[7] П. А. Рыжов, *Математическое определение „геологической ошибки" при подсчёте запасов полезных ископаемых*, Труды Казах. горн.-металл. института 1 (1938) (quoted after [8]).

[8] В. И. Смирнов, *Подсчёт запасов минерального сырья*, Москва 1950.

[9] R. M. Williams, *The variance of the mean of systematic samples*, Biometrika 43 (1956), p. 137-148.

[10] S. Zubrzycki, *O szacowaniu parametrów złóż geologicznych (On estimating gangue parameters)*, Zastosowania Matematyki 3 (1957), p. 105-153.

[11] — *Szacowanie złóż jako zagadnienie statystyczne (Estimating ore-gangues as a statistical problem)*, to be published in Przegląd Geologiczny.

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REMARKS ON COMPACT SEMIGROUPS

BY

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The purpose of this note is to show that some theorems concerning Hausdorff compact (= bicompact) semigroups can be proved by a unified method based on a very simple topological lemma.

Theorem 1, Corollary to Theorem 2 and Theorem 2a are known; we bring them into our considerations to emphasize the method presented in the paper and for the convenience of the reader, since they are needed in the rest of the paper. The remaining results seem to be new, though probably they may be partly known to those who are occupied with compact semigroups. The results of section 1 and a part of the results of section 2 are closely connected with some results stated in a paper of Wallace [1] (see also Wallace [2]).

1. We shall be concerned with functions f of k variables running over a set X and with values in the same set X. We shall write $f(\overset{k}{\breve{X}})$ or $f^{(1)}(\overset{k}{\breve{X}})$ instead of $f(\underbrace{X\times\ldots\times X}_{k \text{ times}})$ and $f^{(n)}(\overset{k}{\breve{X}})$ instead of $f[f^{(n-1)}(\overset{k}{\breve{X}})\times\ldots\times f^{(n-1)}(\overset{k}{\breve{X}})]$.

LEMMA 1. *Let X be a compact space and $f(x_1, \ldots, x_k)$ a continuous function for which $f(\overset{k}{\breve{X}}) \subset X$. Let X_ξ, $\xi < a$, be a decreasing family of non--empty closed subsets of X. Suppose that for each $\xi_0 < a$ there exist ξ_1, ξ_2, $\xi_0 \leqslant \xi_1 \leqslant \xi_2 < a$ such that $f(\overset{k}{\breve{X}}_{\xi_1}) = X_{\xi_2}$. Then $f(\bigcap_{\xi<a} \overset{k}{\breve{X}}_\xi) = \bigcap_{\xi<a} X_\xi$.*

Proof. Let φ_ξ, ψ_ξ be two ordered sets of ordinal numbers both confinal in a and such that $f(\overset{k}{\breve{X}}_{\varphi_\xi}) = X_{\psi_\xi}$. (The existence of such sequences follows from the suppositions.) Since X_ξ is a decreasing family we have

$$\bigcap_{\xi<a} X_\xi = \bigcap_{\xi<a} X_{\varphi_\xi} = \bigcap_{\xi<a} X_{\psi_\xi} \neq \emptyset.$$

If each of the elements $x_1, \ldots, x_k$ is in $\bigcap_\xi X_\xi$, then $f(x_1, \ldots, x_k)$ is contained in every X_{φ_ξ}, hence $f(x_1, \ldots, x_k) \in \bigcap_\xi X_{\varphi_\xi} = \bigcap_\xi X_\xi$. This proves that $f(\bigcap_\xi \overset{k}{\check{X}}_\xi) \subset \bigcap_\xi X_\xi$.

Suppose on the other hand that $x \in \bigcap_\xi X_\xi$. Then for every ξ the element x belongs to X_{φ_ξ}. Hence $X_{\varphi_\xi} \times \ldots \times X_{\varphi_\xi} \cap f^{-1}(x)$ is for every ξ a non-empty closed set. From the compactness of X we get

$$[\bigcap_\xi X_{\varphi_\xi} \times \ldots \times \bigcap_\xi X_{\varphi_\xi}] \cap f^{-1}(x) = \bigcap_\xi [X_{\varphi_\xi} \times \ldots \times X_{\varphi_\xi}] \cap f^{-1}(x) \neq \emptyset.$$

Hence there exist $y_1, \ldots, y_k$ such that $[y_1, \ldots, y_k] \in \bigcap_\xi X_{\varphi_\xi} \times \ldots \times \bigcap_\xi X_{\varphi_\xi} \cap f^{-1}(x)$ for every ξ. Then we have $f(y_1, \ldots, y_k) = x$ and $y_i \in \bigcap_\xi X_{\varphi_\xi} = \bigcap_\xi X_\xi$ for $i = 1, 2, \ldots, k$. This proves that $\bigcap_\xi X_\xi \subset f(\bigcap_\xi \overset{k}{\check{X}}_\xi)$.

In what follows we shall often need the following special case of Lemma 1:

LEMMA 2. *Let X be a compact space, A a non-empty closed subset of X and f a continuous function of k variables with $f(\overset{k}{\check{A}}) \subset A$. Write $T = \bigcap_{n=1}^{\infty} f^{(n)}(\overset{k}{\check{A}}) = f^{(\omega_0)}(\overset{k}{\check{A}})$. Then $f(\overset{k}{\check{T}}) = T$.*

2. THEOREM 1. *Let S be a compact semigroup. Put $T = \bigcap_{i=1}^{\infty} S^i$. Then $T \neq \emptyset$ and $T^2 = T$.*

Proof. Consider the function of two variables $f(x_1, x_2) = x_1 \cdot x_2$. We have $f(\overset{2}{\check{S}}) = S^2$ and $f^{(n)}(\overset{2}{\check{S}}) = S^{2^n}$ for every $n \geqslant 1$. Further $f^{(\omega_0)}(\overset{2}{\check{S}}) = \bigcap_{n=1}^{\infty} S^{2^n} = \bigcap_{i=1}^{\infty} S^i = T \neq \emptyset$. Lemma 2 implies $f(\overset{2}{\check{T}}) = T$, i. e. $T^2 = T$, q. e. d.

THEOREM 2. *Let S be a compact semigroup and A a closed subsemigroup of S. For every $a \in A$ there exists a unique maximal subsemigroup $R^{(a)}(A) \subset A$ with the property $aR^{(a)}(A) = R^{(a)}(A)$. The set $R^{(a)}(A)$ is at the same time a closed right ideal of A.*

Proof. Put $f(x) = ax$ and $f^{(\omega_0)}(A) = \bigcap_{i=1}^{\infty} a^i A = R^{(a)}(A)$. Lemma 2 implies $f[R^{(a)}(A)] = aR^{(a)}(A) = R^{(a)}(A)$. Obviously $R^{(a)}(A)$ is a closed right ideal of A. It remains to show that $R^{(a)}(A)$ is the greatest subsemigroup $A_1 \subset A$ with the property $aA_1 = A_1$. Let A_1 be any subsemigroup with this property. Then $A_1 = aA_1 \subset aA$. This relation implies $A_1 = aA_1 \subset a^2 A$ and so on. Hence $A_1 \subset \bigcap_{i=1}^{\infty} a^i A = R^{(a)}(A)$. This proves our assertion.

Remark. Suppose in Theorem 2 that A is the closure of the set $Z_a = \{a, a^2, a^3, \ldots\}$. Clearly $\bar{Z}_a$ is a commutative semigroup. Put, for brevity, $R^{(a)}(\bar{Z}_a) = G^{(a)}$. Then $G^{(a)} \subset \bar{Z}_a$ and $aG^{(a)} = G^{(a)}$ implies $a^n G^{(a)} = G^{(a)}$ for every $n \geqslant 1$. It is elementary to prove that the set of all $t \in S$ with $tG^{(a)} = G^{(a)}$ is closed. Since $\bar{Z}_a$ is the smallest closed set containing $\{a, a^2, \ldots\}$, we conclude that for every $b \in \bar{Z}_a$ (and moreover for every $b \in G^{(a)}$) $bG^{(a)} = G^{(a)}$ holds. This shows that $G^{(a)}$ is a group. Hence $\bar{Z}_a$ contains an idempotent e_a, namely the unity element of $G^{(a)}$. Since $G^{(a)} \subset \bar{Z}_a$, we have $G^{(a)} = G^{(a)} e_a \subset \bar{Z}_a e_a \subset \bar{Z}_a G^{(a)} = \bigcup_{b \in \bar{Z}_a} bG^{(a)} = G^{(a)}$. This implies $G^{(a)} = \bar{Z}_a e_a$. Finally, if $\bar{Z}_a$ contained an idempotent $e \neq e_a$, either $e \in \bar{Z}_a - a\bar{Z}_a \neq \emptyset$ would hold or there would be an integer $n \geqslant 1$ with $e \in a^n \bar{Z}_a - a^{n+1} \bar{Z}_a \neq \emptyset$. The first possibility implies $e = a$, whence $e = e_a$, contrarily to the supposition. In the second case we should have $e = a^n u$, $u \in \bar{Z}_a$. But then $e = e^2 = a^{2n} u^2 \in a^{n+1} \bar{Z}_a$, which is a contradiction. We have proved the following known result:

COROLLARY. *If S is compact, the closure of the set Z_a contains a unique idempotent e_a. The set $\bar{Z}_a \cdot e = G^{(a)}$ is a group, further a minimal ideal of $\bar{Z}_a$ and the greatest subsemigroup G of $\bar{Z}_a$ with the property $aG = G$.*

If e_a is the idempotent belonging to $\bar{Z}_a$, we shall say in what follows that a belongs to the idempotent e_a.

THEOREM 2a. *Let the suppositions of Theorem 2 be satisfied. If a belongs to e_a, then $R^{(a)}(A) = e_a A$.*

Proof. The relation $aR^{(a)}(A) = R^{(a)}(A)$ implies $a^n R^{(a)}(A) = R^{(a)}(A)$ for every $n \geqslant 1$. The same argument as in the last Remark shows that for every $b \in \bar{Z}_a$ we have $bR^{(a)}(A) = R^{(a)}(A)$, especially $e_a R^{(a)}(A) = R^{(a)}(A)$. Since $a \in A$, we have $\bar{Z}_a \subset A$ and $\bigcap_{i=1}^{\infty} a^i \bar{Z}_a \subset \bigcap_{i=1}^{\infty} a^i A$, i. e. $G^{(a)} \subset R^{(a)}(A)$, especially $e_a \in R^{(a)}(A)$. Now we have $R^{(a)}(A) = e_a R^{(a)}(A) \subset e_a A \subset R^{(a)}(A) \cdot A \subset R^{(a)}(A)$. This proves $R^{(a)}(A) = e_a A$.

In the special case $A = S$ we shall write $R^{(a)}$ instead of $R^{(a)}(S)$ and get:

THEOREM 2b. *Let S be a compact semigroup. For every $a \in S$ there exists a unique maximal subsemigroup $R^{(a)}$ having the property $aR^{(a)} = R^{(a)}$. The set $R^{(a)}$ is a closed right ideal of S, and if a belongs to e_a we have $R^{(a)} = e_a S$.*

Remark. Needless to say that there is a left dual version of this theorem. For every $a \in S$ there exists a closed subsemigroup $L^{(a)}$ with $L^{(a)} \cdot a = L^{(a)}$ and $L^{(a)}$ is maximal with respect to the required property. The set $L^{(a)}$ is a left ideal of S and if a belongs to e_a, we have $L^{(a)} = Se_a$.

THEOREM 3. *Let S be a compact semigroup. For every $a \in S$ there exists a unique maximal subsemigroup $M^{(a)}$ with the property $M^{(a)}aM^{(a)} = M^{(a)}$. The set $M^{(a)}$ is a closed two-sided ideal of S and $M^{(a)} \supset L^{(a)}R^{(a)}$ holds.*

Proof. Let us put $f(x, y) = xay$. Then $f(\overset{2}{\check{S}}) = SaS$. In general for every $n \geqslant 1$ we have $f^{(n)}(S \times S) = (Sa)^{2^n-1}S = S(aS)^{2^n-1}$. Put

$$f^{(\omega_0)}(S \times S) = \bigcap_{n=1}^{\infty} (Sa)^{2^n-1}S = M^{(a)}.$$

Clearly $M^{(a)}$ is a closed two-sided ideal of S. Lemma 2 implies $f(M^{(a)} \times M^{(a)}) = M^{(a)}$, i. e. $M^{(a)}aM^{(a)} = M^{(a)}$. If S_1 is any subsemigroup with $S_1aS_1 = S_1$, then $S_1 = S_1aS_1 \subset SaS$. This relation implies $S_1 = S_1aS_1 \subset (SaS)a(SaS)$, i. e. $S_1 \subset (Sa)^3S$. Repeating this argument we get $S_1 \subset (Sa)^{2^n-1}S$ for every $n \geqslant 1$. Hence $S_1 \subset M^{(a)}$. This proves the statement concerning the maximality of $M^{(a)}$.

For use in what follows, note first that for any idempotent $e \in S$ we have $Se \supset (Se)(Se) \supset S \cdot e \cdot e \cdot e = Se$, whence $(Se)^2 = Se$. Suppose now that a belongs to e_a. Since $e_a \in Sa$, we have $Se_a \subset Sa$ and $Se_a = (Se_a)^n \subset (Sa)^n$ for all $n \geqslant 1$. Therefore $Se_a \subset \bigcap_{n=1}^{\infty} (Sa)^n$ and $L^{(a)}R^{(a)} = Se_a \cdot e_aS = Se_aS \subset \bigcap_{n=1}^{\infty} (Sa)^nS = M^{(a)}$. This completes the proof of Theorem 3.

Remark 1. In general it is not true that $L^{(a)}R^{(a)} = M^{(a)}$. This can be shown on the following example. Let $S = \{0, b_1, b_2, b_3, b_4\}$ be a semigroup with the following multiplication table:

	0	b_1	b_2	b_3	b_4
0	0	0	0	0	0
b_1	0	b_1	0	b_3	0
b_2	0	0	b_2	0	b_4
b_3	0	0	b_3	0	b_1
b_4	0	b_4	0	b_2	0

We have $L^{(b_3)} = \{0\}$, $R^{(b_3)} = \{0\}$, further $L^{(b_3)}R^{(b_3)} = \{0\}$, but $M^{(b_3)} = Sb_3S = S$.

Remark 2. Call an element $a \in S$ *regular*, if it is contained in a subgroup of S. If a belongs to e_a and a is regular, then a is contained in the maximal group $H(e_a)$ belonging to e_a. Then there is an $a' \in H(e_a)$ with $aa' = a'a = e_a$. We also have $aa'a = a$ and $Sa \supset (Sa)(Sa) \supset Saa'a = Sa$, whence $(Sa)^2 = Sa$ and $M^{(a)} = SaS$. Since $Se_a = Sa'a \subset S^2a \subset Sa = Saa'a \subset S^2a'a = S^2e_a \subset Se_a$, we have $Se_a = Sa$. Therefore $M^{(a)} = SaS = Se_aS = (Se_a)(e_aS) = L^{(a)}R^{(a)}$. We have proved:

COROLLARY. *For every regular $a \in S$ we have $M^{(a)} = L^{(a)}R^{(a)}$.*

Remark 3. The same relation can easily be proved for every $a \in S$, if S is commutative.

THEOREM 4. *Let S be a compact semigroup. For every $a \in S$ there exists a unique maximal subsemigroup $P^{(a)}$ with the property $aP^{(a)}a = P^{(a)}$. The semigroup $P^{(a)}$ is closed and $R^{(a)}L^{(a)} = P^{(a)}$ holds.*

Proof. Let us put $f(x) = axa$. Then $f(S) = aSa$ and $f^{(n)}(S) = a^nSa^n$. The set $f^{(\omega_0)}(S) = \bigcap_{n=1}^{\infty} a^nSa^n = P^{(a)} \neq \emptyset$ is a closed subsemigroup and Lemma 2 yields $aP^{(a)}a = P^{(a)}$.

If S_1 is any subsemigroup with $S_1 = aS_1a$, then $S_1 \subset aSa$. This implies $S_1 = aS_1a \subset a(aSa)a = a^2Sa^2$ and so on. Hence $S_1 \subset P^{(a)}$, which proves that $P^{(a)}$ is the greatest subsemigroup with the required property.

The relation $aP^{(a)}a = P^{(a)}$ implies $a^nP^{(a)}a^n = P^{(a)}$ for every integer $n \geqslant 1$. If a belongs to e_a we conclude (as above) that $e_aP^{(a)}e_a = P^{(a)}$. Note that $e_aS^2e_a \subset e_aSe_a = (e_aS)e_a \subset e_aS \cdot Se_a$, whence $e_aS^2e_a = e_aSe_a$ and therefore $R^{(a)}L^{(a)} = e_aS \cdot Se_a = e_aSe_a$. Now, since $aR^{(a)}L^{(a)}a = R^{(a)}L^{(a)}$ implies $a(e_aSe_a)a = e_aSe_a$ and $P^{(a)}$ is the greatest subsemigroup with this property, we necessarily have $e_aSe_a \subset P^{(a)}$. On the other hand we have $e_aSe_a \supset e_aP^{(a)}e_a = P^{(a)}$. Hence $P^{(a)} = e_aSe_a = R_{(a)}L^{(a)}$, q. e. d.

Remark. Since $L^{(a)}[R^{(a)}]$ is a left [right] ideal of S, we clearly have $R^{(a)}L^{(a)} \subset R^{(a)} \cap L^{(a)}$. On the other hand, if $z \in R^{(a)} \cap L^{(a)}$, there are elements $u, v \in S$ such that $z = ue_a = e_av$. This implies $z = e_aze_a$, i. e. $z \in e_aSe_a = R^{(a)}L^{(a)}$, hence $R^{(a)}L^{(a)} = R^{(a)} \cap L^{(a)}$.

To sum up, we have proved the following "inequalities", which in general cannot be strengthened:

$$P^{(a)} = R^{(a)} \cap L^{(a)} = R^{(a)}L^{(a)} \begin{matrix} \subset R^{(a)} \\ \subset L^{(a)} \end{matrix} \subset L^{(a)}R^{(a)} \subset M^{(a)}.$$

3. In this section we consider another kind of functions $f(x)$.

THEOREM 5. *Let S be a compact semigroup and T_k the set of those $a \in S$ which have roots of every degree k^n $(n = 1, 2, 3, \ldots)$. Then T_k is a closed non-empty subset of S and every $a \in T_k$ has a root of degree k in T_k.*

Proof. Put $f_k(x) = x^k$ and $f_k^{(\omega_0)}(S) = T_k$. Lemma 2 yields $f_k(T_k) = T_k$. The last relation implies that for every $b \in T_k$ there is an $a \in T_k$ with $f_k(a) = a^k = b$.

Theorem 6. *Let S be a compact semigroup and T the set of those $a \epsilon S$ which have in S roots of every degree $n = 1, 2, 3, \ldots$ Then T is a closed non-empty subset of S such that every $a \epsilon T$ has roots of every degree in T.*

Proof. If k/l, we clearly have $T_k \supset T_l$. Next we prove $f_k(T_l) = T_l$. If $c \epsilon T_l$, then $c^\tau \epsilon T_l$ for every integer $\tau \geqslant 1$, especially $c^k \epsilon T_l$, therefore $f_k(T_l) \subset T_l$. Suppose conversely that $c \epsilon T_l$. Then according to Theorem 5 there is a $d \epsilon T_l$ with $c = d^l = (d^{l/k})^k$. Since $d \epsilon T_l$ implies $d^{l/k} \epsilon T_l$, we have $c \epsilon f_k(T_l)$, whence $T_l \subset f_k(T_l)$. This proves our assertion. Now the sequence $T_{1!} \supset T_{2!} \supset T_{3!} \supset \ldots$ fulfils the suppositions of Lemma 1. Hence $T = \bigcap_{n=1}^{\infty} T_{n!} \neq \emptyset$ and $f_k(T) = T$ for every $k > 0$. This proves Theorem 6.

If S is commutative, T is obviously a semigroup. Let us call a semigroup U *complete* if every element from U has in U roots of every degree $k > 0$. We then have:

Theorem 7. *In a compact commutative semigroup S the set of elements having roots of every degree $k > 0$ forms a complete closed subsemigroup.*

This theorem is known for compact abelian groups (T is then a group) and need not hold for discrete groups.

We use this opportunity to prove a further theorem on (non-necessarily commutative) complete compact semigroups.

Theorem 8. *Let S be a complete compact semigroup. Let e be an idempotent from S and $H(e)$ the maximal group belonging to e. Then $H(e)$ is a complete closed group.*

Proof. Let $a \epsilon H(e)$. Then according to the supposition there is an $x = x(n) \epsilon S$ with $x^n = a$ for every $n \geqslant 1$. Let $n \geqslant 1$ be arbitrary, but fixed. We know that the set $\overline{\{x, x^2, x^3, \ldots\}}$ contains a unique idempotent. Since $\overline{\{x, x^2, x^3, \ldots\}} \supset \overline{\{x^n, x^{2n}, x^{3n}, \ldots\}} = \overline{\{a, a^2, a^3, \ldots\}}$ and a belongs to e, this idempotent is e. Hence x belongs to e. Now every element $x \epsilon S$ belonging to e satisfies $xe = ex \epsilon G^{(x)} \subset H(e)$. Since $a = x^n$, we have $ea = ex^n$, whence $a = (ex)^n$, i. e. $a = y^n$ with $y \epsilon H(e)$. We have proved that for every n there is a $y = y(n) \epsilon H(e)$ such that $y^n = a$. This proves that $H(e)$ is a complete group. The fact that $H(e)$ is closed is well known.

REFERENCES

[1] A. D. Wallace, *Inverses in Euclidean mobs*, Mathematical Journal of Okayama University 3 (1953), p. 23-28.

[2] — *The structure of topological semigroups*, Bulletin of the American Mathematical Society 61 (1955). p. 95-112.

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ON FINITE COCYCLES AND THE SPHERE THEOREM

BY

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1. Introduction. Let M be a closed (i. e. compact, unbounded), connected 3-manifold. We describe a 2-sphere in M as *tame* if, and only if, it is the image of $S \times 1/2$ in a homeomorphism of $S \times I$ into M, where S is a 2-sphere. We describe M as *reducible* if, and only if, it contains a tame 2-sphere, S, which is essential in M (i. e. the identical map $S \to M$ is not homotopic to a constant). In this case M may be "reduced" by cutting through S and filling in the holes. If M is reducible, then (cf. § 6 below; also [3]) $\pi_1(M)$ is either cyclic infinite or a free product with two non-trivial factors. In fact we shall prove (cf. [3]):

Theorem (1.1). *For M to be reducible it is necessary and sufficient that $\pi_1(M)$ be either cyclic infinite or a non-trivial free product.*

We emphasize the fact that M need not be orientable. Specker [9] has proved that $\pi_2(M)$ is a free Abelian group whose rank is $0, 1$ or ∞ according as $\pi_1(M)$ has less than 2, 2 or ∞ ends [2]. If $\pi_1(M)$ is cyclic infinite or a non-trivial free product, it has 2 or ∞ ends. Therefore if M is orientable (1.1) follows from the triangulation theorem [4] and the sphere theorem [6, 10].

In order to prove (1.1) we consider a certain Π-module $J(\Pi, G)$, which is associated with a given group Π and a given Abelian group G (see § 5 below). We write $J(\Pi, Z) = J(\Pi)$, where Z is the group of integers. According to Specker [9] there is an operator isomorphism $J(\pi_1(M)) \approx \pi_2(M)$. Assuming that Π is finitely presentable, we introduce a certain sub-set $\Sigma(\Pi, G) \subset J(\Pi, G)$. In general $\Sigma(\Pi, G)$ is not a sub-group of $J(\Pi, G)$ but it contains the element 0. If $\Pi = 1$ or $G = 0$, then $J(\Pi, G) = 0$. We write $\Sigma(\Pi, Z) = \Sigma(\Pi)$. We shall prove:

Theorem (1.2). *In order that a finitely presentable group, Π, be either cyclic infinite or a non-trivial free product it is necessary and sufficient that $\Sigma(\Pi, G) \neq 0$ for a given $G \neq 0$.*

Corollary (1.3). *If $\Sigma(\Pi, G) \neq 0$, then $\Sigma(\Pi, G') \neq 0$, where G' is any non-zero Abelian group.*