COLLOQUIUM MATHEMATICUM

VOL. VI DÉDIÉ À M. CASIMIR KURATOWSKI 1958

THE CHARACTERISTIC EQUATION FOR PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

BY

A. PLIŚ (CRACOW)

Let the solution $z(x, y)$ of the partial differential equation

$$z_x = f(x, y, z, z_y), \tag{1}$$

and the function $f(x, y, z, q)$ be of class C^1.

DEFINITION 1. *A solution $w(x)$ of the ordinary differential equation*

$$w' = -f_q(x, w, z(x, w), z_y(x, w)) \tag{2}$$

will be called a basic one for (1) *if the functions $y(x) = w(x)$, $z(x) = z(x, w(x))$, $q(x) = z_y(x, w(x))$ satisfy the system of ordinary differential equations*

$$\begin{cases} y' = -f_q(x, y, z, q), \\ z' = f(x, y, z, q) - qf_q(x, y, z, q), \\ q' = f_y(x, y, z, q) + qf_z(x, y, z, q). \end{cases} \tag{3}$$

DEFINITION 2. *We say that Property P is satisfied for the functions f, z if all solutions of* (2) *are basic for* (1).

In paper [2] (theorem 1) a necessary and sufficient condition (valid for two or more independent variables) was given for Property P to be satisfied. That condition is imposed both on the function f and on z.

In this paper we give a sufficient condition for Property P to be satisfied that is imposed on the function f only (Theorem T).

LEMMA. *If the function $k(x)$ is continuous for $a \leqslant x \leqslant b$ $(a < b)$ and $(k(b) - k(a))/(b-a) > m$, then there exist two numbers c, d, $a \leqslant c < d \leqslant b$, such that*

$$k(c) + m(x-c) \leqslant k(x) \leqslant k(d) + m(x-d) \quad \text{for} \quad c \leqslant x \leqslant d. \tag{4}$$

Analogically if $(k(b) - k(a))/(b-a) < m$, then there exist two numbers c^*, d^*, $a \leqslant c^* < d^* \leqslant b$, such that

$$k(c^*) + m(x-c^*) \geqslant k(x) \geqslant k(d^*) + m(x-d^*) \quad \text{for} \quad c^* \leqslant x \leqslant d^*.$$

Proof. We take

$$c = \max x\{x\colon a \leqslant x < b,\ k(x) = k(a)+m(x-a)\},$$

$$d = \min x\{x\colon c < x \leqslant b,\ k(x) = k(b)+m(x-b)\}.$$

It is easy to verify inequalities (4). The numbers c^*, d^* may be defined in a similar manner.

THEOREM T. *Suppose that the function $f(x, y, z, q)$ defined in an (open) set N is of class C^1 and satisfies in N the following properties:*

$$(5)\qquad |f_q(x, y, z, q)-f_q(x, y^*, z^*, q)| \leqslant L(|y-y^*|+|z-z^*|),$$

where L is a constant, and

$$(6)\qquad f_q(x, y, z, q) \geqslant f_q(x, y, z, q^*) \quad \textit{for} \quad q \geqslant q^*.$$

Under the assumptions above Property P is satisfied for the function f and each solution $z(x, y)$ of (1) defined in an open set and of class C^1.

Proof. Consider a solution $z(x, y)$ of (1) defined in an open set D and of class C^1. Let $w(x)$ be an arbitrary solution of (2). It is evident that the functions $y(x) = w(x)$, $z(x) = z(x, w(x))$, $q = k(x) = z_y(x, w(x))$ satisfy the first two equations (3). For the proof of Theorem T we must show that the last equation (3) is also satisfied. Suppose the contrary. Then there exists a number s, $(s, w(s))\in D$, for which the last equation (3) is not satisfied. Write

$$g = f_y(s, w(s), z(s), k(s))+k(s)\cdot f_z(s, w(s), z(s), k(s)).$$

Hence there exist a positive number R and a sequence s_n, $s_n \to s$, such that

$$(7)\qquad |(k(s_n)-k(s))/(s_n-s)-g| > R > 0 \qquad (n = 1, 2, \ldots).$$

Let $\theta\colon t_1 \leqslant x \leqslant t_2 (t_1 < s < t_2)$ be such an interval that every integral $H^{(t)}\colon w = h^{(t)}(x)$ of (2) passing through the point $(t, w(t))$, where $t_1 \leqslant t \leqslant t_2$, can be enlarged on θ, $(x, h^{(t)}(x))\in D$ for $t_1 \leqslant x \leqslant t_2$ $(t_1 \leqslant t \leqslant t_2)$ and the inequality

$$(8)\qquad |f_y(x, y, z(x, y), z_y(x, y))+z_y(x, y)f_z(x, y, z(x, y), z_y(x, y))-g| < R$$

is satisfied on all the integrals $H^{(t)}$ for $t_1 \leqslant x \leqslant t_2$, $t_1 \leqslant t \leqslant t_2$.

Let us fix a number s_n from interval θ. It follows from (7) that either $(k(s_n)-k(s))/(s_n-s) > g+R$, or $(k(s_n)-k(s))/(s_n-s) < g-R$.

We consider only the first case because the second one is quite analogous. From the Lemma it follows that there exists such an interval $[c, d]$ $(c < d)$, contained in the interval $[s_n, s]$, that

$$(9)\qquad k(x) \geqslant k(c)+(g+R)(x-c) \quad \text{for} \quad c \leqslant x \leqslant d$$

and $k(x) \leqslant k(d)+(g+R)(x-d)$ for $c \leqslant x \leqslant d$.

It follows from [1], p. 419, that through each point (ξ, η) of D there passes at least one basic solution $w^*(x)$ of (2) defined in a neighbourhood of ξ. It easily follows that there exist basic solutions of (2), $u(x)$, $v(x)$, satisfying the conditions

$$(10)\qquad u(c) = w(c), \quad v(d) = w(d)$$

respectively and defined in the interval $[c, d]$.

Now we prove that

$$(11)\qquad u(x) \geqslant w(x), \quad v(x) \geqslant w(x) \quad \text{for} \quad c \leqslant x \leqslant d.$$

Write $q^{\square}(x) = z_y(x, u(x))$. From (3) and (8) follows the inequality $q^{\square}(x) \leqslant q^{\square}(c)+(g+R)(x-c)$. By (10) $q^{\square}(c) = k(c)$. Hence, by (9), we have

$$(12)\qquad z_y(x, u(x)) \leqslant k(x) \quad \text{for} \quad c \leqslant x \leqslant d.$$

The function $z(x, y)$ being of class C^1 we have

$$(13)\qquad |z(x, w(x))-z(x, u(x))| \leqslant K|w(x)-u(x)| \quad \text{for} \quad c \leqslant x \leqslant d,$$

where K is a constant.

The function $r(x) = w(x)-u(x)$ fulfils the identity

$$r'(x) = f_q(x, u(x), z(x, u(x)), z_y(x, u(x)))-f_q(x, w(x), z(x, w(x)), k(x)).$$

It follows from (12) and (6) that

$$r'(x) \leqslant f_q(x, u(x), z(x, u(x)), k(x))-f_q(x, w(x), z(x, w(x)), k(x)).$$

By (5) and (13) the differential inequality $r'(x) \leqslant L(1+K)r(x)$ $(c \leqslant x \leqslant d)$ is satisfied if $r(x) > 0$.

From the theorem on differential inequalities (see for instance [3], p. 3) by the equality $r(c) = 0$ it follows that the inequality $r(x) \leqslant 0$, i. e. the first inequality (11), is satisfied for $c \leqslant x \leqslant d$. In a similar manner the second inequality (11) can be proved.

It follows from (10) and (11) that there exists a number e, $c \leqslant e \leqslant d$, such that $u(e) = v(e)$. Along the basic solution $y = \lambda(x)$, where $\lambda(x) = u(x)$ for $c \leqslant x \leqslant e$ and $\lambda(x) = v(x)$ for $e \leqslant x \leqslant d$, inequality (8) being satisfied, we have by (3) the inequality

$$k(d)-k(c) = z_y(d, \lambda(d))-z_y(c, \lambda(c)) < (g+R)(d-c).$$

From (9) it follows that $k(d)-k(c) \geqslant (g+R)(d-c)$. This contradiction completes the proof.

Remark 1. The example in [2] shows that assumption (6) cannot be omitted also for f being a polynomial. It may be replaced by the assumption that $f_q(x, y, z, q) \leqslant f_q(x, y, z, q^*)$ for $q \geqslant q^*$.

Remark 2. The assumptions of Theorem T are satisfied in particular for $f = a(x, y, z,)q + b(x, y, z)$, where a, b are of class C^1.

REFERENCES

[1] A. Pliś, *Characteristics of non-linear partial differential equations*, Bulletin de l'Académie Polonaise des Sciences, Classe III, 2 (1954), p. 419.

[2] — *On characteristics of partial differential equations*, ibidem 5 (1957), p. 957.

[3] T. Ważewski, *Certaines propositions de caractère „épidermique" relatives aux inégalités différentielles*, Annales de la Société Polonaise de Mathématiques 24 (1951), p. 1-12.

Reçu par la Rédaction le 3. 2. 1958

COLLOQUIUM MATHEMATICUM

VOL. VI DÉDIÉ À M. CASIMIR KURATOWSKI 1958

SUR LES DÉCOMPOSITIONS D'ENSEMBLES CONNEXES

PAR

B. KNASTER, A. LELEK ET JAN MYCIELSKI (WROCŁAW)

I. Introduction

Nous entendons dans cette communication par *décomposition* d'un ensemble de points celle en une somme de ses sous-ensembles non-vides et disjoints.

La *connexité* d'un ensemble est, par définition, son indécomposabilité en deux ensembles fermés dans lui [1]. Les *continus* sont les ensembles à la fois connexes et compacts ou — ce qui est la même chose dans les espaces compacts — les fermetures d'ensembles connexes. Les ensembles connexes et ouverts s'appellent *régions*.

L'indécomposabilité en deux ensembles fermés entraînant celle en n'importe quel nombre fini n d'ensembles fermés (puisque la somme de $n-1$ ensembles de ce genre en est encore un), la question s'impose si un X connexe peut se décomposer en une suite dénombrable d'ensembles fermés dans lui. Cette question a été résolue d'abord pour les continus: aucun continu X ne se laisse décomposer ainsi [2], mais il existe, déjà sur le plan, des X connexes et localement compacts, en particulier des ensembles connexes et fermés (dits aussi *continus non-bornés*) qui se décomposent en suites dénombrables d'ensembles fermés dans eux [3]; pourtant, au moins l'un d'eux ne peut alors être connexe [4]. Il existe aussi parmi les F_σ plans bornés, mais qui ne sont pas des G_δ (donc, à plus forte raison, localement compacts), qui se laissent dé-

[1] Voir [5], p. 303 et [1] p. 244.

[2] Voir [9], p. 300.

[3] Voir [2], p. 58 et [10], p. 5.

[4] Voir [6]. Ce travail contient aussi un exemple du continu non-borné irréductible entre deux points, décomposable en un seul ensemble fermé non-connexe et une infinité dénombrable de continus non-bornés; cf. également [7].