Pour $p = 8$, nous avons pu construire par ce procédé une surface de genres $p_a = p_g = 0$, $P_2 = 3$, $P_3 = 7$ (voir [11]).

8. Nous avons désigné par $r_1, r_2, \ldots r_{p-1}$ les dimensions des systèmes linéaires $|L_1|, |L_2|, \ldots, |L_{p-1}|$. D'après la théorie des homographies cycliques, on a

$$r_1+r_2+\ldots+r_{p-1}+p-1 = p_g.$$

Si la surface F est régulière, cette relation prend la forme

$$r_1+r_2+\ldots+r_{p-1}+p-1 = p_a = p-1,$$

d'où $r_1 = r_2 = \ldots = r_{p-1} = 0$. Donc, si la surface F est régulière, les courbes $K_1, K_2, \ldots, K_{p-1}$ sont isolées.

Inversement, si ces courbes sont isolées, on a $p_g = p-1 = p_a$ et F est régulière.

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POISSON DISTRIBUTIONS ON COMPACT ABELIAN TOPOLOGICAL GROUPS

BY

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1. Let G be a compact Abelian topological group. A regular completely additive measure μ defined on the class of all Borel subsets of G, with $\mu(G) = 1$, will be called a *probability distribution*. Let X_1, X_2 be the pair of independent G-valued random variables with the probability distributions μ_1, μ_2. Let us denote by λ the probability distribution of the random variable $X_1 \cdot X_2$, where the product is taken in the sense of group multiplication in G.

It is well known that $\lambda = \mu_1 * \mu_2$, where the convolution $*$ is defined by the formula

$$\mu_1 * \mu_2(E) = \int_G \mu_1(Ex^{-1})\mu_2(dx).$$

We say that a probability distribution μ is a *Poisson distribution with the parameter* x_0 $(x_0 \in G)$ if there exists a non-negative constant m such that

$$\mu(E) = \sum_{k \in K(E)} \frac{m^k}{k!} \exp(-m), \tag{1}$$

where $K(E)$ denotes the set of all indices k for which $x_0^k \in E$.

We say that a probability distribution μ is a *composed Poisson distribution* if there exists a regular completely additive measure ν defined on the class of all Borel subsets of G, with $\nu(G) < \infty$, such that

$$\mu = \sum_{k=0}^{\infty} \frac{\nu^{*k}}{k!} \exp(-\nu(G)), \tag{2}$$

where

$$\nu^{*0}(E) = \begin{cases} 1 & \text{if } e \in E, \\ 0 & \text{if } e \notin E, \end{cases}$$

$$\nu^{*(k+1)} = \nu * \nu^{*k} \quad (k = 0, 1, \ldots),$$

and e denotes the unit element of G. It is easy to see that a composed Poisson distribution is a Poisson distribution if and only if $\nu = m\delta_{x_0}$, where

$$\delta_{x_0}(E) = \begin{cases} 1 & \text{if} \quad x_0 \in E, \\ 0 & \text{if} \quad x_0 \notin E. \end{cases}$$

In the present paper we shall give the following characterization of composed Poisson distributions and Poisson distributions:

THEOREM 1. *A probability distribution μ is a composed Poisson distribution if and only if there exists a sequence of probability distributions $\mu_1, \mu_2, \ldots$ such that*

(3) $$\mu = \mu_n^{*n} \quad (n = 1, 2, \ldots)$$

and

(4) $$\lim_{n\to\infty} \mu_n(e) = 1.$$

THEOREM 2. *A probability distribution μ is a Poisson distribution with the parameter x_0 or*

$$\mu(e) = \mu(x_0) = \tfrac{1}{2}, \quad \mu(E) = 0 \quad \textit{for} \quad E \cap (e \cup x_0) = 0$$

if and only if there exists a sequence of probability distributions $\mu_1, \mu_2, \ldots$ such that $\mu = \mu_n^{*n}$ $(n = 1, 2, \ldots)$, $\liminf_{n\to\infty} \mu_n(e) > 0$ and

(5) $$\lim_{n\to\infty} n\mu_n\big(G \setminus (e \cup x_0)\big) = 0.$$

Moreover, if $x_0^2 \neq e$ or $x_0 = e$, then μ is a Poisson distribution.

We remark that the results of this paper are known for finite Abelian groups (cf. [5]).

II. Before proving the Theorems we shall give some elementary properties of the characteristic function of the probability distribution. $\hat{G}$ will denote the group of all continuous characters of the group G (cf. [4], Chapter IV). The function

$$\varphi_\mu(\chi) = \int_G \chi(x)\mu(dx) \quad (\chi \in \hat{G})$$

is called the *characteristic function of the probability distribution* μ. It is easy to prove that

(6) $$\varphi_{\mu_1 * \mu_2}(\chi) = \varphi_{\mu_1}(\chi)\cdot\varphi_{\mu_2}(\chi).$$

Let $\mathcal{B}$ be the Banach space of all continuous complex-valued functions f in G with the norm $\|f\| = \max_{x \in G} |f(x)|$. By $\mathcal{B}_0$ we shall denote the subspace of $\mathcal{B}$ containing all functions vanishing at e. According to the Theorem of Peter-Weyl (cf. [4], § 21,22) every function belonging to $\mathcal{B}$ can be uniformly approximated by linear combinations of characters. Hence the equality $\varphi_\mu(\chi) = \varphi_\lambda(\chi)$ for $\chi \in \hat{G}$ implies

$$\int_G f(x)\mu(dx) = \int_G f(x)\lambda(dx) \quad \text{for} \quad f \in \mathcal{B},$$

and consequently $\mu = \lambda$. Thus the probability distribution is uniquely determined by the characteristic function.

It is easy to prove that the characteristic function of the composed Poisson distribution (2) has the form

$$\varphi_\mu(\chi) = \exp \int_G \big(\chi(x) - 1\big)\nu(dx).$$

In particular, the characteristic function of the Poisson distribution (1) has the form $\varphi_\mu(\chi) = \exp m\big(\chi(x_0) - 1\big)$.

A probability distribution λ is called *symmetric* if it is invariant under the transformation $x \to x^{-1}$, *i. e.* if $\lambda(E) = \lambda(E^{-1})$ for each Borel subset $E \subset G$, where $E^{-1} = \{x^{-1}: x \in E\}$. It is easy to prove that λ is a symmetric probability distribution if and only if φ_λ is a real-valued function.

LEMMA 1. *Let $\mu_1, \mu_2, \ldots$ be a sequence of probability distributions and $\lim_{n\to\infty} \mu_n(e) = 1$. Then $\lim_{n\to\infty} \varphi_{\mu_n}(\chi) = 1$ uniformly for $\chi \in \hat{G}$.*

Proof. The assertion of the Lemma is a direct consequence of the following inequality:

$$|1 - \varphi_{\mu_n}(\chi)| = \Big|\int_{G \setminus e} \big(1 - \chi(x)\big)\mu_n(dx)\Big| \leqslant 2\big(1 - \mu_n(e)\big).$$

LEMMA 2. *Let $\lambda_1, \lambda_2, \ldots$ be a sequence of symmetric probability distributions such that*

(7) $$\lambda_1 = \lambda_n^{*n} \quad (n = 1, 2, \ldots)$$

and

(8) $$\lim_{n\to\infty} \lambda_n(e) = 1.$$

Then $\sup_{n \geqslant 1} \sup_{\chi \in \hat{G}} n\big(1 - \varphi_{\lambda_n}(\chi)\big) < \infty.$

Proof. Since φ_{λ_n} $(n = 1, 2, \ldots)$ are real-valued functions, then, according to (6), assumption (7) implies

$$\varphi_{\lambda_n}(\chi) = \sqrt[n]{\varphi_{\lambda_1}(\chi)} \qquad (n = 1, 2, \ldots).$$

Hence and from the assumption (8), in view of Lemma 1, it follows that there is an index n_0 such that $\frac{1}{2} \leqslant \varphi_{\lambda_n}(\chi) \leqslant 1$ for $n \geqslant n_0$, $\chi \in \hat{G}$. Consequently,

$$1/2^{n_0} \leqslant \varphi_{\lambda_1}(\chi) \leqslant 1 \quad \text{for} \quad \chi \in \hat{G}.$$

Hence the sequence

$$n\big(1-\varphi_{\lambda_n}(\chi)\big) = n\big(1-\sqrt[n]{\varphi_{\lambda_1}(\chi)}\big) \qquad (n = 1, 2, \ldots)$$

converges to $-\log\varphi_{\lambda_1}(\chi)$ uniformly for $\chi \in \hat{G}$, which implies the assertion of Lemma.

LEMMA 3. *Let $\lambda_1, \lambda_2, \ldots$ be a sequence of symmetric probability distributions. Suppose that conditions* (7) *and* (8) *are fulfilled. Then* $\sup_{n\geqslant 1} n\lambda_n(G\setminus e) < \infty$.

Proof. By m we shall denote the Haar measure of G normalized by supposing $m(G) = 1$. Let U $(U \subset G)$ be an arbitrary neighbourhood of e. There is then a neighbourhood V such that

$$V\cdot V^{-1} \subset U. \tag{9}$$

It is well known that there exists a continuous function f_V which vanishes off V and

$$\int_G f_V(x)m(dx) \neq 0. \tag{10}$$

Put

$$g_V(x) = \int_G f_V(y)\overline{f_V(yx^{-1})}m(dy). \tag{11}$$

The function g_V is continuous on G and the equality $f_V(x) = 0$ $(x \notin V)$ implies

$$g_V(x) = \int_V f_V(y)\overline{f_V(yx^{-1})}m(dy). \tag{12}$$

For each $y \in V$ we have the equality $f_V(yx^{-1}) = 0$ if $x \notin V\cdot V^{-1}$. Consequently, according to (12),

$$g_V(x) = 0 \quad \text{if} \quad x \notin V\cdot V^{-1}. \tag{13}$$

Since g_V is the convolution of functions with m-integrable squares, the Fourier expansion

$$g_V(x) = \sum_{\chi\in\hat{G}} c_V(\chi)\chi(x), \tag{14}$$

where

$$c_V(\chi) = \int_G g_V(y)\overline{\chi(y)}m(dy),$$

converges uniformly for $x \in G$ (cf. [4], § 22). Obviously, $c_V(\chi) = 0$ except on a countable set of characters. From formulas (10) and (11) it follows that

$$c_V(\chi) = \Big|\int_G f_V(y)\overline{\chi(y)}m(dy)\Big|^2 \qquad (\chi \in \hat{G}) \tag{15}$$

and

$$0 < \sum_{\chi\in\hat{G}} c_V(\chi) = g_V(e) < \infty. \tag{16}$$

According to Lemma 2, there is a positive constant M such that

$$n\int_G \big(1-\chi(x)\big)\lambda_n(dx) \leqslant M$$

for each $\chi \in \hat{G}$ and each $n \geqslant 1$. Since λ_n $(n = 1, 2, \ldots)$ are symmetric, the last inequality implies

$$n\int_{G\setminus V\cdot V^{-1}} \big(1-\chi(x)\big)\lambda_n(dx) \leqslant M$$

for each $\chi \in \hat{G}$ and each $n \geqslant 1$. Hence, in view of (15) and (16),

$$n\int_{G\setminus V\cdot V^{-1}} \Big(\sum_{\chi\in\hat{G}} c_V(\chi) - \sum_{\chi\in\hat{G}} c_V(\chi)\chi(x)\Big)\lambda_n(dx) \leqslant M\sum_{\chi\in\hat{G}} c_V(\chi)$$

for $n = 1, 2, \ldots$ Hence, according to (14),

$$n\int_{G\setminus V\cdot V^{-1}} \big(g_V(e) - g_V(x)\big)\lambda_n(dx) \leqslant Mg_V(e) \qquad (n = 1, 2, \ldots)$$

Taking into account the formulas (13) and (16) we have $n\lambda_n(G\setminus V\cdot V)^{-1} \leqslant M$ $(n = 1, 2, \ldots)$. Hence, according to (9), for every neighbourhood U of the unit element e the inequality $n\lambda_n(G\setminus U) \leqslant M$ $(n = 1, 2, \ldots)$ is true. Consequently, $n\lambda_n(G\setminus e) \leqslant M$ $(n = 1, 2, \ldots)$.

The Lemma is thus proved.

Proof of Theorem 1. *Sufficiency of conditions* (3) *and* (4). Suppose that conditions (3) and (4) are satisfied. Put

$$\bar{\mu}_n(E) = \mu_n(E^{-1}) \qquad (n = 1, 2, \ldots)$$

and

(17) $$\lambda_n = \mu_n * \bar{\mu}_n \qquad (n = 1, 2, \ldots).$$

It is easy to verify that λ_n $(n = 1, 2, \ldots)$ are symmetric probability distributions and

$$\lambda_1 = \lambda_n^{*n} \qquad (n = 1, 2, \ldots),$$

(18) $$\lambda_n(E) \geqslant \mu_n(E)\mu_n(e) \qquad (n = 1, 2, \ldots)$$

for each Borel subset $E \subset G$. The last inequality, in virtue of the assumption (4), implies $\lim_{n\to\infty} \lambda_n(e) = 1$. Consequently, according to Lemma 3, $\sup_{n\geqslant 1} n\lambda_n(G\setminus e) < \infty$. Hence, in view of (4) and (18),

(19) $$\sup_{n\geqslant 1} n\mu_n(G\setminus e) \leqslant \sup_{n\geqslant 1} \frac{n\lambda_n(G\setminus e)}{\mu_n(e)} < \infty.$$

Let $\mathfrak{B}_0^*$ be the conjugate space of $\mathfrak{B}_0$, *i.e.* the space of all continuous linear functionals on $\mathfrak{B}_0$. Put

(20) $$L_n(f) = n\int_G f(x)\mu_n(dx) \qquad (n = 1, 2, \ldots; f \in \mathfrak{B}_0).$$

Then $|L_n(f)| \leqslant \|f\| n\mu_n(G\setminus e)$ $(n = 1, 2, \ldots; f \in \mathfrak{B}_0)$. Consequently, according to (19),

(21) $$\sup_{n\geqslant 1} \|L_n\| < \infty.$$

Let us consider the weak topology in $\mathfrak{B}_0^*$, *i. e.* the topology generated by the family of neighbourhoods of 0

$$U(f_1, f_2, \ldots, f_n; \varepsilon) = \bigcap_{k=1}^{n} \{L: |L(f_k)| < \varepsilon\},$$

where ε is an arbitrary positive number and $f_k \in \mathfrak{B}_0$ $(1 \leqslant k \leqslant n)$. Since the strongly closed sphere in $\mathfrak{B}_0^*$ is compact in the weak topology (cf. [2], p. 22), in view of (21) the weak closure A_n of the set $\{L_k : k \geqslant n\}$ is weakly compact. Since $A_n \supset A_{n+1}$ and $A_n \neq 0$ $(n = 1, 2, \ldots)$, there is a linear functional L_∞ such that

$$L_\infty \in \bigcap_{n=1}^{\infty} A_n.$$

From the definition of the weak topology it follows that for every $f \in \mathfrak{B}_0$ there exists a sequence of indices $k_1 < k_2 < \ldots$ such that

(22) $$L_\infty(f) = \lim_{n\to\infty} L_{k_n}(f).$$

Since $L_n(f) \geqslant 0$ for $f(x) \geqslant 0$ $(x \in G, f \in \mathfrak{B}_0)$, the last equality implies $L_\infty(f) \geqslant 0$ for $f(x) \geqslant 0$ $(x \in G, f \in \mathfrak{B}_0)$. Consequently, there is a regular completely additive measure ν defined on the class of all Borel subsets of G, with $\nu(G) < \infty$ (cf. [1], p. 247 and 248) such that

$$L_\infty(f) = \int_G f(x)\nu(dx) \qquad (f \in \mathfrak{B}_0).$$

Hence, in view of (20) and (22), it follows that for every $f \in \mathfrak{B}_0$ there is a sequence of indices $k_1 < k_2 < \ldots$ such that

$$\lim_{n\to\infty} k_n \int_G f(x)\mu_{k_n}(dx) = \int_G f(x)\nu(dx).$$

Let $\chi \in \hat{G}$. Then the function $\chi(x) - 1 = \chi(x) - \chi(e)$ belongs to $\mathfrak{B}_0$. Consequently

(23) $$\lim_{n\to\infty} k_n \int_G \big(\chi(x)-1\big)\mu_{k_n}(dx) = \int_G \big(\chi(x)-1\big)\nu(dx)$$

for a sequence of indices $k_1 < k_2 < \ldots$ From equalities (3) and (6) it follows that

$$\varphi_\mu(\chi) = \big(\varphi_{\mu_n}(\chi)\big)^n = \left(1 + \frac{n\int_G \big(\chi(x)-1\big)\mu_n(dx)}{n}\right)^n \qquad (n = 1, 2, \ldots).$$

Hence, according to (23),

$$\varphi_\mu(\chi) = \exp\int_G \big(\chi(x)-1\big)\nu(dx) \qquad (\chi \in \hat{G}).$$

Thus μ is a composed Poisson distribution.

Necessity of conditions (3) *and* (4). Suppose that μ is a composed Poisson distribution and equality (2) holds. Put

(24) $$\mu_n = \sum_{k=0}^{\infty} \frac{\nu^{*k}}{k!\,n^k} \exp\left(-\frac{\nu(G)}{n}\right) \qquad (n = 1, 2, \ldots).$$

It is easy to verify that equality (3) holds. Further, we have

$$\mu_n(e) \geqslant \nu^{*0}(e)\exp\left(-\frac{\nu(G)}{n}\right) = \exp\left(-\frac{\nu(G)}{n}\right) \qquad (n = 1, 2, \ldots),$$

which implies equality (4). The Theorem is thus proved.

LEMMA 4. *Let $\mu_1, \mu_2, \ldots$ be a sequence of probability distributions satisfying the conditions*

(25) $$\mu_1 = \mu_n^{*n} \quad (n = 1, 2, \ldots),$$

(26) $$\liminf_{n\to\infty} \mu_n(e) > 0,$$

(27) $$\lim_{n\to\infty} \mu_n(G\setminus(e\cup x_0)) = 0$$

for some $x_0 \in G$.

Then $\mu_n(E) = \mu_n(Ex_0)$ $(n = 1, 2, \ldots)$ for each Borel subset $E \subset G$ or $\lim_{n\to\infty}\mu_n(e) = 1$.

Moreover, if $x_0^2 \neq e$ or $x_0 = e$, then the last equality holds.

Proof. Let λ_n $(n = 1, 2, \ldots)$ be the sequence of symmetric probability distribution defined by formula (17). Then

(28) $$\varphi_{\lambda_n}(\chi) = |\varphi_{\mu_n}(\chi)|^2 = \sqrt[n]{|\varphi_{\mu_1}(\chi)|^2} \quad (n = 1, 2, \ldots; \chi \in \hat{G}).$$

From definition (17) it follows that

$$\lambda_n(G\setminus(e\cup x_0\cup x_0^{-1})) = \int_{G\setminus(e\cup x_0)} \mu_n(Gx\setminus(e\cup x_0\cup x_0^{-1})x)\mu_n(dx) + \mu_n(G\setminus(x_0\cup x_0^2\cup e))\mu_n(x_0) + \mu_n(G\setminus(e\cup x_0\cup x_0^{-1}))\mu_n(e) \leqslant 3\mu_n(G\setminus(e\cup x_0)) \quad (n = 1, 2, \ldots).$$

Hence, in view of (27), we obtain

(29) $$\lim_{n\to\infty} \lambda_n(G\setminus(e\cup x_0\cup x_0^{-1})) = 0.$$

From equality (28) it follows that the limit

(30) $$\psi(\chi) = \lim_{n\to\infty} \varphi_{\lambda_n}(\chi) \quad (\chi \in \hat{G})$$

exists and $(\psi(\chi))^2 = \psi(\chi)$. Consequently, there is a closed subgroup G_0 of G such that

(31) $$\psi(\chi) = \varphi_{m_0}(\chi) \quad (\chi \in \hat{G}),$$

where m_0 is the Haar measure of the subgroup G_0 normalized so that $m_0(G_0) = 1$ and $m_0(E) = m_0(E\cap G_0)$ for each Borel subset E of G (see [3], p. 259). Hence, in view of (29) and (30),

(32) $$G_0 \subset e\cup x_0\cup x_0^{-1}.$$

First we suppose that

(33) $$G_0 = \{e\}.$$

Then, according to (30) and (31),

$$\lim_{n\to\infty} \varphi_{\lambda_n}(\chi) = \chi(e) = 1 \quad (\chi \in \hat{G}).$$

Hence, in view of (28),

(34) $$\lim_{n\to\infty} |\varphi_{\mu_n}(\chi)| = 1 \quad (\chi \in \hat{G}).$$

Further, we have, in virtue of (27),

$$\lim_{n\to\infty} \int_{G\setminus(e\cup x_0)} \chi(x)\mu_n(dx) = 0 \quad (\chi \in \hat{G}).$$

Hence and from (34) we obtain for $x_0 \neq e$

(35) $$\lim_{n\to\infty} |\chi(x_0)\mu_n(x_0) + \mu_n(e)| = 1 \quad (\chi \in \hat{G}).$$

It is well known that for $x_0 \neq e$ there exists a character χ_0 such that $\chi_0(x_0) \neq 1$ (cf. [4], § 27). Equality (35) for $\chi = \chi_0$ implies

$$\lim_{n\to\infty} \min(\mu_n(x_0), \mu_n(e)) = 0.$$

Hence, in virtue of (26) and (27),

$$\lim_{n\to\infty} \mu_n(e) = 1 \quad \text{for} \quad x_0 \neq e.$$

Since for $x_0 = e$ the last equality is a direct consequence of (26) and (27), we obtain the assertion of the Lemma in the case (33).

Now we assume that $G_0 = \{e, x_0, x_0^{-1}\}$ and $x_0 \neq e$.

Since m_0 is the Haar measure of G_0, then $\varphi_{m_0}(\chi) = 0$ if $\chi(x_0) \neq 1$ $(\chi \in \hat{G})$ (cf. [4], § 20). Hence, according to (28), (30) and (31), $\varphi_{\mu_n}(\chi) = 0$ if $\chi(x_0) \neq 1$. This implies

$$\int_G \chi(x)\mu_n(x_0 dx) = \overline{\chi(x_0)} \int_G \chi(x)\mu_n(dx) = \int_G \chi(x)\mu_n(dx) \quad (n = 1, 2, \ldots; \chi \in \hat{G}).$$

Consequently, for every Borel subset E of G the equality

(36) $$\mu_n(E) = \mu_n(Ex_0) \quad (n = 1, 2, \ldots)$$

holds.

Let $x_0^2 \neq e$. Then $x_0^2 \neq x_0$, and, according to (27), $\lim_{n\to\infty}\mu_n(x_0^2) = 0$.

From equality (36) it follows that $\mu_n(x_0) = \mu_n(x_0^2)$ $(n = 1, 2, \ldots)$. Consequently, $\lim_{n\to\infty}\mu_n(x_0) = 0$, which, in virtue of (27), implies the relation $\lim_{n\to\infty}\mu_n(e) = 1$.

The Lemma is thus proved.

Proof of Theorem 2. *Sufficiency*. Suppose that the probability distribution μ satisfies the conditions of the Theorem. From Lemma 4 it follows that

$$\lim_{n\to\infty}\mu_n(e) = 1 \tag{37}$$

or

$$\mu_n(E) = \mu_n(Ex_0) \quad (n = 1, 2, \ldots) \tag{38}$$

for all Borel subset E of G. Moreover, equality (37) holds if $x_0^2 \neq e$ or $x_0 = e$.

First we consider the case (38) for $x_0 \neq e$, $x_0^2 = e$. Since $G_0 = \{e, x_0\}$ is the compact subgroup of G, then the quotient group G/G_0 is compact. Further, if F is a Borel subset of G/G_0, then $F \cup Fx_0$ is a Borel subset of G. Put

$$\tilde{\mu}_n(F) = \mu_n(F\cup Fx_0) \quad (n = 1, 2, \ldots). \tag{39}$$

It is easy to verify, in view of (38), that $\tilde{\mu}_n$ $(n = 1, 2, \ldots)$ are probability distributions on G/G_0 and

$$\tilde{\mu}_1 = \tilde{\mu}_n^{*n} \quad (n = 1, 2, \ldots). \tag{40}$$

By $\tilde{e}$ we shall denote the unit element of G/G_0. From equality (39) it follows that $\tilde{\mu}_n(G/G_0\setminus\tilde{e}) = \mu_n\big(G\setminus(e\cup x_0)\big)$ $(n = 1, 2, \ldots)$. Consequently, according to (5),

$$\lim_{n\to\infty} n\tilde{\mu}_n(G/G_0\setminus\tilde{e}) = 0. \tag{41}$$

Hence and from (40), in virtue of Theorem 1, we infer that $\tilde{\mu}_1$ is a composed Poisson distribution on G/G_0. There is then a regular completely additive measure $\tilde{\nu}$ defined on the class of all Borel subsets of G/G_0, with $\tilde{\nu}(G/G_0) < \infty$, such that the characteristic function $\varphi_{\tilde{\mu}_1}$ is given by the following formula:

$$\varphi_{\tilde{\mu}_1}(\chi) = \exp\int_{G/G_0}\big(\chi(x)-1\big)\tilde{\nu}(dx) \quad (\chi\in\widehat{G/G_0}). \tag{42}$$

Put $\tilde{\lambda}_n = \tilde{\mu}_n * \bar{\tilde{\mu}}_n$ $(n = 1, 2, \ldots)$. Then, according to (40) and (42),

$$\varphi_{\tilde{\lambda}_n}(\chi) = \sqrt[n]{|\varphi_{\tilde{\mu}_1}(\chi)|^2} = \exp\int_{G/G_0}\big(\chi(x)-1\big)\tilde{\nu}_n(dx),$$

where

$$\tilde{\nu}_n(F) = \frac{\tilde{\nu}(F)+\tilde{\nu}(F^{-1})}{n} \quad (n = 1, 2, \ldots).$$

Consequently

$$\tilde{\lambda}_n = \sum_{k=0}^{\infty}\frac{\tilde{\nu}_n^{*k}}{k!}\exp\big(-\tilde{\nu}_n(G/G_0)\big) \quad (n = 1, 2, \ldots). \tag{43}$$

Since

$$\tilde{\lambda}_n(G/G_0\setminus\tilde{e}) = \int_{G/G_0}\tilde{\mu}_n\big((G/G_0\setminus\tilde{e})x\big)\tilde{\mu}_n(dx) \leqslant$$

$$\leqslant \tilde{\mu}_n(G/G_0\setminus\tilde{e})\big(1+\tilde{\mu}_n(\tilde{e})\big) \quad (n = 1, 2, \ldots),$$

equality (41) implies

$$\lim_{n\to\infty} n\tilde{\lambda}_n(G/G_0\setminus\tilde{e}) = 0. \tag{44}$$

From equality (43) it follows that

$$n\tilde{\lambda}_n(G/G_0\setminus\tilde{e})\exp\big(\tilde{\nu}_n(G/G_0)\big) \geqslant n\tilde{\nu}_n(G/G_0\setminus\tilde{e}) = 2\tilde{\nu}(G/G_0\setminus\tilde{e}).$$

Taking into account equality (44) we obtain $\tilde{\nu}(G/G_0\setminus\tilde{e}) = 0$. Hence, according to (42), $\varphi_{\tilde{\mu}_1}(\chi) \equiv 1$ $(\chi\in\widehat{G/G_0})$, which implies $\tilde{\mu}_1(\tilde{e}) = 1$. This equality, in view of (39), implies $\mu(e\cup x_0) = \tilde{\mu}_1(\tilde{e}) = 1$. Hence, according to (38) and the assumption $x_0 \neq e$, we obtain $\mu(e) = \mu(x_0) = \frac{1}{2}$.

In the case (38) for $x_0 \neq e$, $x_0^2 = e$ the Theorem is thus proved.

For the other case we have equality (37). Let $\lambda_1, \lambda_2, \ldots$ be the sequence of symmetric probability distributions defined by the formula $\lambda_n = \mu_n * \bar{\mu}_n$ $(n = 1, 2, \ldots)$. Then, according to (37), $\lim_{n\to\infty}\lambda_n(e) = 1$. Hence, in view of Lemma 3, $\sup_{n\geqslant 1} n\lambda_n(G\setminus e) < \infty$. which implies

$$\sup_{n\geqslant 1} n\mu_n(G\setminus e) < \infty. \tag{45}$$

To prove this we must reason in the same way as in the proof of Theorem 1.

Let $x_0 \neq e$. From inequality (45) it follows that there is a sequence of indices $k_1 < k_2 < \ldots$ for which the limit

$$m = \lim_{n\to\infty} k_n\mu_{k_n}(x_0)$$

exists. Setting $m = 0$ for $x_0 = e$ we obtain, in virtue of (5), for each $\chi\in\hat{G}$

$$\lim_{n\to\infty} k_n\int_G\big(\chi(x)-1\big)\mu_{k_n}(dx) = m\big(\chi(x_0)-1\big).$$

Since

$$\varphi_\mu(\chi) = \left((\varphi_{\mu_{k_n}}(\chi)\right)^{k_n} = \left(1 + \frac{k_n \int\limits_G (\chi(x)-1)\mu_{k_n}(dx)}{k_n}\right)^{k_n}$$

we have $\varphi_\mu(\chi) = \exp m(\chi(x_0)-1)$.

Thus μ is a Poisson distribution with the parameter x_0.

Necessity. First we suppose that μ is a Poisson distribution and equality (1) holds. Let μ_n $(n = 1, 2, \ldots)$ be defined by formula (2.1) with $\nu = m\delta_{x_0}$. Then

$$\mu = \mu_n^{*n} \quad (n = 1, 2, \ldots), \qquad \lim_{n\to\infty} \mu_n(e) = 1$$

and

$$\mu_n(G \setminus (e \cup x_0)) \leqslant 1 - \exp\left(-\frac{m}{n}\right) - \frac{m}{n}\exp\left(-\frac{m}{n}\right) \qquad (n = 1, 2, \ldots).$$

Consequently $\lim\limits_{n\to\infty} n\mu_n(G \setminus (e \cup x_0)) = 0$

Now we assume that $x_0^2 = e$, $x_0 \neq e$ and $\mu(e) = u(x_0) = \frac{1}{2}$. Setting $\mu_n = \mu$ $(n = 1, 2, \ldots)$ we have

$$\mu = \mu_n^{*n}, \qquad \mu_n(e) = \tfrac{1}{2} \quad \text{and} \quad \mu_n(G \setminus (e \cup x_0)) = 0 \qquad (n = 1, 2, \ldots).$$

The Theorem is thus proved.

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CONCERNING APPROXIMATION WITH NODES

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This note contains a remark on the subject treated by Paszkowski [1], [2].

Define

$$E_n = \min_{P_n(x)} \max_{-1\leqslant x\leqslant 1} |f(x) - P_n(x)|, \qquad E_n' = \min_{P_n(0)=f(0)} \max_{-1\leqslant x\leqslant 1} |f(x) - P_n(x)|$$

where $P_n(x)$ runs through all polynomials of degree n. Clearly

$$E_n \leqslant E_n' \leqslant 2E_n. \tag{1}$$

I shall prove that there exists an $f(x)$ satisfying

$$\overline{\lim_{n=\infty}} E_n'/E_n = 2. \tag{2}$$

Let $n_k \to \infty$ sufficiently fast. Put

$$f(x) = \sum_{k=1}^{\infty} T_{2n_k}(x)/k!,$$

where $T_n(x)$ is the n-th Tchebycheff polynomial. Because of $|T_{2n}(0)| = 1$ we have

$$E_{2n_k} \leqslant (1+o(1))/(k+1)! \qquad \left(P_n(x) = \sum_{j=1}^{k} T_{2n_j}(x)/j!\right). \tag{3}$$

Next we show that

$$E_{2n_k}' \geqslant (2+o(1))/(k+1)!. \tag{4}$$

Equality (2) follows from (1), (3) and (4). Thus we only have to show (4).

Let $\Theta_{2n_k}(x)$ be the polynomial of degree $\leqslant 2n_k$ for which

$$\max_{-1\leqslant x\leqslant 1} |f(x) - \Theta_{2n_k}(x)| = E_{2n_k}'.$$