THÉORÈME. *Si l'on a $\nu(n)=\nu(n+1)=1$, ou bien on a $n=8$, ou bien n est un nombre premier de Mersenne, ou bien $n+1$ est un nombre premier de Fermat.*

D'autre part, dans chacun de ces trois cas, on a évidemment $\nu(n) = \nu(n+1) = 1$. On ne connaît actuellement que cinq nombres premiers de Fermat: 3, 5, 17, 257 et 65537, qui donnent les valeurs $n = 2, 4, 16$, 256 et 65536, et on sait qu'il n'existe aucun autre nombre premier de Fermat $< 2^{2^{13}}+1$. D'autre part, on connaît les nombres premiers de Mersenne consécutifs jusqu'au 17-ème, qui est égal à $2^{2281}-1 < 2^{2^{13}}+1$. Ainsi, nous pouvons affirmer que nous connaissons tous les nombres naturels $n < 2^{2281}$ tels que $\nu(n) = \nu(n+1) = 1$ et qu'il y en a 23. Le 24-ème, qu'on connaît encore en ce moment est le nombre premier de Mersenne $2^{3217}-1$.

On pourrait démontrer sans peine qu'il n'existe que trois nombres naturels n, à savoir 2, 3 et 7, tels que $\nu(n) = \nu(n+1) = \nu(n+2) = 1$, qu'il existe un seul nombre $n=2$ tel que $\nu(n) = \nu(n+1) = \nu(n+2) = \nu(n+3) = 1$ et, par conséquent, qu'il n'existe aucun nombre naturel n tel que $\nu(n) = \nu(n+1) = \nu(n+2) = \nu(n+3) = \nu(n+4) = 1$.

La question reste ouverte (**P 252**) s'il existe une infinité de nombres naturels n tels que $\nu(n) = \nu(n+1)$. Or il résulte d'une hypothèse de A. Schinzel sur les nombers premiers (voir [3] ou [4]) l'existence d'une infinité de nombres naturels n tels que $\nu(n) = \nu(n+1) = \nu(n+2) = 2$, et aussi l'existence, pour tout nombre naturel k, d'une infinité de nombres naturels n tels que $\nu(n) = \nu(n+1) = \ldots = \nu(n+k)$.

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ON THE NUMBER OF LATTICE POINTS INSIDE A CLOSED CURVE

BY

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1. Results. We establish a rectangular system of coordinates axes XY on the plane. Each set isometric with

$$S = \{(x, y): x, y \text{ are integers}\}$$

will be called an *integral lattice*. The plane Lebesgue measure of a set E will be denoted by $|E|$. Let G be an open and bounded set. We suppose that the boundary L of G is a (simple) closed curve and that $|L| = 0$. In this paper we shall prove the following

THEOREM. *If $|G|$ is an integer, then there exists such an integral lattice S_0 that the set $S_0 \cap G$ counts $|G|$ elements.*

Let O be the origin of our coordinates system. For any set E and for each point p let us denote by E_p (by E_{-p}) the translation of E defined by the vector $\overrightarrow{Op}$ ($\overrightarrow{pO}$). Let $f(p)$ be the number of elements of the set $S_p \cap G$. Let us denote by m, M the lowest and the greatest values of $f(p)$ which are attained on sets of positive measures. In the second section we shall prove

LEMMA 1. *It is $m \leqslant |G| \leqslant M$ and if*

(*) $\bigcup_S (L_p \cap L_q)$, *where the summation runs over all different $p, q \in S$, contains no subcontinua,*

then $f(p)$ attains all integral values between m and M.

In the third section we shall prove

LEMMA 2. *There exists an integral lattice S_0 for which* (*) *holds.*

It is evident that these lemmas imply the theorem.

Remark. The above theorem as well as these lemmas remain true if L is a finite sum of closed curves.

2. Proof of Lemma 1. Let p be a point. We shall denote the set $\{p\}$ also by p. The sets

$$\{q: q \in S, p_q \in G\}, \qquad \{q: q \in S, p \in G_{-q}\}$$

have the same number of elements since $p_q \epsilon G \equiv p \epsilon G_{-q}$. The first of them counts evidently $f(p)$ elements. If we denote by $\chi_q(p)$ the characteristic function of G_{-q}, then the number of elements of the second set is $\sum_{q\epsilon S}\chi_q(p)$. So we obtain

$$(1)\qquad f(p) = \sum_{q\epsilon S}\chi_q(p).$$

We define $Q = \{(x, y): 0 \leqslant x, y \leqslant 1\}$. Then evidently

$$(2)\qquad m \leqslant \int_Q f(p)\,dp \leqslant M.$$

From (1) follows

$$(3)\qquad \int_Q f(p)\,dp = \sum_{q\epsilon S}\int_Q \chi_q(p)\,dp = \sum_{q\epsilon S}\int_{Q_q}\chi_0(p)\,dp = |G|$$

(here the sums are finite since G is bounded) [1]. From (2) and (3) follows $m \leqslant |G| \leqslant M$.

Let us now prove that the assumption that Lemma 1 is not true yields a contradiction. For this aim we suppose that some integer k satisfies $m \leqslant k \leqslant M$ and $f(p) \neq k$ for each p. Thus the sets $A = \{p: f(p) < k\}$ and $B = \{p: f(p) > k\}$, which are of course both of non zero measure, cover the plane. It follows from well known results that the set $\bigcup_S (L_p \cap L_q)$ where the summation runs as in (*) does not cut the plane, i. e. its complementary C is a connected set [2]. Since the measure of $\bigcup_S (L_p \cap L_q)$ is zero we obtain that $A \cap C$ and $B \cap C$ are not empty. Consequently there exists a point $r \epsilon C$ which is common to their closures. Evidently we can find such neighbourhood W of r which intersects at most one of the sets L_q where $q \epsilon S$. Then for $p \epsilon W$ at most one point of the lattice S_p, namely p_{-q}, lies on L. Thus the function $f(p)$ takes on W one or two values which are consecutive integers. We arrived to a contradiction since $W \cap A \cap C$ and $W \cap B \cap C$ are not empty sets.

3. Proof of Lemma 2. Let us denote by S_φ the set obtained from S by a rotation around O through the angle φ. For two sets A, B we shall write

$$A \underset{\varphi}{\sim} B$$

if B is a translation of A given by a vector $\overrightarrow{Op}$ where $p \epsilon S_\varphi$. It is evident that (*) holds for S_φ if and only if no arcs $L', L'' \subset L$ satisfy $L' \underset{\varphi}{\sim} L''$. Such angle φ exists since

(**) *The set of all angles φ for which there exist such arcs $L', L'' \subset L$ that $L' \underset{\varphi}{\sim} L''$ is at most countable.*

So it remains to prove (**). Let us denote by W a denumerable set of arcs such that each arc $L' \subset L$ contains an arc which belongs to W. Evidently it is sufficient to prove (**) under the assumption that $L' \epsilon W$. Let us observe that (**) follows from the assertions

I. *For any two arcs $L', L'' \subset L$ there exists only a finite number of such φ that $L' \underset{\varphi}{\sim} L''$.*

II. *For any arc $L' \epsilon W$ the set of all arcs $L'' \subset L$ for which there exist such φ that $L' \underset{\varphi}{\sim} L''$ holds is finite.*

Proof of I. Let $\overrightarrow{Op}$ be the vector which defines the translation transforming L' on L''. Evidently there exist only a finite number of integral lattices S_φ for which $p \epsilon S_\varphi$.

Proof of II. We suppose that L is a directed curve so that any two points $u, v \epsilon L$ define an arc (u, v) of L with the initial point u and the endpoint v. We shall write

$$(u, v) \approx (p, v)$$

if $p \epsilon (u, v)$ and if for the translation T defined by $Tp = u$ it is $T(p, v) = (u, T(v))$. Let us observe the following property (P_1) of $\approx$:

(P_1) *If $(u, v) \approx (p, v)$, $T(p, v) = (u, Tv)$ and $q \epsilon (u, Tv)$, $(u, v) \approx (q, v)$, then $(u, v) \approx (T^{-1}q, v)$.*

Let us denote by Ω the set of all p which satisfy $(u, v) \approx (p, v)$. We shall say that p is a *left limit point* of a set E if in each arc (p, q) are points of E (different from p). Then it is

(P_2) *If u is a left limit point of Ω then $\Omega = (u, v)$ and (u, v) is a segment.*

Let us prove (P_2). From (P_1) follows that each point of Ω is a left limit point of Ω. Now Ω is a closed set and thus if some point $q \epsilon (u, v)$ does not belong to Ω then there exists such an arc $(a, b) \subset (u, v)$ that $(a, b) \cap \Omega = \{a\}$. This is impossible since a is a left limit point of Ω.

If there exists a point $q \epsilon (u, v)$ which does not belong to the segment $[u, v]$, then there exists such an arc $(a, b) \subset (u, v)$ that $(a, b) \cap [u, v] = \{a\}$. Thus a is a left limit point of such points q for which the distance $d(q)$ from the straight line uv is greater than $d(a)$. From $(u, v) = \Omega$ follows that each point $p \epsilon (u, v)$ which lies on the same side of the line uv as (a, b) has also the above property of a. This is in contradiction with the existence of such points p which lie on the same side of uv as (a, b) and on which $d(p)$ attains its maximum.

[1] Formula (3) implies $f(p) \geqslant |G|$ for some p. This is the so called *Blichfeldt's theorem*.

[2] C. Kuratowski, *Topologie II*, Warszawa 1952, p. 335.

Let us suppose now that II does not hold. Then there exist an arc $L' = (p, q)$ and a sequence of arcs (u_i, v_i) such that $(p, q) \underset{\varphi_i}{\sim} (u_i, v_i)$. For each i one of the vectors $\overrightarrow{pu_i}, \overrightarrow{pv_i}$ is the translation vector and thus it has the length $\sqrt{m}$ where m is an integer. Since L is bounded, there is only a finite number of lengths possible for $\overrightarrow{pu_i}$ or $\overrightarrow{pv_i}$. Therefore we may assume (restricting ourselves to a subsequence $\{(u_{i_k}, v_{i_k})\}$) that for some m all points u_i or all points v_i lie on a circle of radius $\sqrt{m}$. We can assume also that there exist

$$\lim_{i\to\infty} u_i = u \quad \text{and} \quad \lim_{i\to\infty} v_i = v.$$

Then, for sufficiently large i, $(u, v) \frown (u_i, v_i) = (u_i, v)$ or (u, v_i).

Since the second of these possibilities can be reduced to the first by changing the sense on L, we can assume that for every i

$$(4) \qquad (u, v) \frown (u_i, v_i) = (u_i, v).$$

Since the arcs (u_i, v_i) are congruent one to another by translations, each of these arcs is congruent by translation to (u, v). Consequently (4) implies that $(u, v) \approx (u_i, v)$ for each i. Thus from (P_2) follows that (u, v) is a segment. But then (u_i, v_i) are segments of a straight line. This is impossible since we obtained that all points u_i or all points v_i lie on a circle. Thus the assumption that II does not hold leads to a contradiction.

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SUR L'EXISTENCE DES INTÉGRALES ASYMPTOTIQUES DES ÉQUATIONS DIFFÉRENTIELLES ISSUES D'UN ENSEMBLE DE DIMENSION ZÉRO

PAR

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§ 1. Nous dirons que l'ensemble Z situé dans l'espace cartésien à trois dimensions E^3 est *du type d'Antoine* lorsque

1° Z est compact et contient au moins deux points différents,

2° Z ne contient aucun sous-ensemble connexe ne se réduisant pas à un point,

3° Si K est un ensemble homéomorphe à la sphère $x^2+y^2+z^2 \leqslant 1$ et si un point de Z est situé à l'intérieur de K tandis qu'un autre point de Z est situé à l'extérieur de K, alors Z possède au moins un point commun avec la frontière de K.

Remarque 1. L'exemple d'un tel ensemble Z est dû à L. Antoine (cf. [1], p. 91-94). La dimension de cet ensemble est égale à zéro.

Remarque 2. On peut facilement démontrer que la projection orthogonale ou oblique d'un ensemble de ce type sur un plan quelconque constitue un continu. On peut pareillement démontrer que la projection „curviligne" de Z, effectuée le long des intégrales d'un système d'équations différentielles de la forme (1) sur une surface simplement connexe (suffisamment régulière et non tangente aux intégrales), constitue un continu.

§ 2. La note présente apporte une condition suffisante pour l'existence des intégrales d'un système de deux équations différentielles,

$$(1) \qquad dx/dt = f(x, y, t), \qquad dy/dt = g(x, y, t),$$

qui commencent sur un ensemble Z du type d'Antoine et possèdent une certaine propriété asymptotique (sont asymptotiques par rapport à un tuyau T). En appliquant la Remarque 2, nous montrons que l'on peut remplacer l'ensemble Z par un ensemble connexe Z_1, ce qui permet d'employer la méthode topologique du travail [2].

En remplaçant dans le théorème le tuyau T par un tuyau convenablement modifié, on peut obtenir des théorèmes relatifs à l'existence