Théorème. Si l'on a \( v(n) = v(n+1) = 1 \), ou bien on a \( n = 8 \), ou bien \( n \) est un nombre premier de Mersenne, ou bien \( n+1 \) est un nombre premier de Fermat.

D'autre part, dans chacun de ces trois cas, on a évidemment \( v(n) = v(n+1) = 1 \). On ne connaît actuellement que cinq nombres premiers de Fermat: 3, 5, 17, 257 et 65537, qui donnent les valeurs \( n = 2, 4, 16, 256 \) et 65535, et on sait qu'il n'existe aucun autre nombre premier de Fermat \( < 2^{32} + 1 \). D'autre part, on connaît les nombres premiers de Mersenne consécutifs jusqu'à un 17-ème, qui est égal à \( 2^{127} - 1 < 2^{32} + 1 \). Ainsi, nous pouvons affirmer que nous connaissons tous les nombres premiers \( n < 2^{20} \) tels que \( v(n) = v(n+1) = 1 \) et qu'il y en a 23. Le 24-ème, qu'on connaît encore en ce moment, est le nombre premier de Mersenne \( 2^{31} - 1 \).

On pourrait démontrer sans peine qu'il n'existe que trois nombres premiers \( n \), à savoir 2, 3 et 7, tels que \( v(n) = v(n+1) = v(n+2) = 1 \), qu'il existe un seul nombre \( n = 2 \) tel que \( v(n) = v(n+1) = v(n+2) = v(n+3) = 1 \) et, par conséquent, qu'il n'existe aucun autre nombre premier \( n \) tel que \( v(n) = v(n+1) = v(n+2) = v(n+3) = v(n+4) = 1 \).

La question reste ouverte (P 252) s'il existe une infinité de nombres premiers \( n \) tels que \( v(n) = v(n+1) \). Or il résulte d'une hypothèse de A. Schnizel sur les nombres premiers (voir [3] ou [4]) l'existence d'une infinité de nombres premiers \( n \) tels que \( v(n) = v(n+1) = v(n+2) = 2 \), et aussi l'existence, pour tout nombre premier \( k \), d'une infinité de nombres premiers \( n \) tels que \( v(n) = v(n+1) = \ldots = v(n+k) \).

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**On the Number of Lattice Points Inside a Closed Curve**

by

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1. Results. We establish a rectangular system of coordinates axes \( XY \) on the plane. Each set isometric with

\[ S = \{(x, y): x, y \text{ are integers}\} \]

will be called an integral lattice. The plane Lebesgue measure of a set \( E \) will be denoted by \( |E| \). Let \( G \) be an open and bounded set. We suppose that the boundary \( L \) of \( G \) is a (simple) closed curve and that \( |L| = 0 \).

In this paper we shall prove the following

Theorem. If \([G]\) is an integer, then there exists such an integral lattice \( S \) that the set \( S \cap G \) counts \([G]\) elements.

Let \( G \) be the origin of our coordinates system. For any set \( E \) and for each point \( p \) let us denote by \( E_p \) (by \( E_{-p} \)) the translation of \( E \) defined by the vector \( p \) (by \( -p \)). Let \( f(p) \) be the number of elements of the set \( S_p \cap G \). Let us denote by \( m \), \( M \) the lowest and the greatest values of \( f(p) \) which are attained on sets of positive measures. In the second section we shall prove

**Lemma 1.** It is \( m \leq |G| \leq M \) and if

\[ \left( \bigcup (L_p \cap L_q) \right) \]

where the summation runs over all different \( p, q \), \( S \)

contains no subintervals,

then \( (G) \) attains all integral values between \( m \) and \( M \).

\( (*) \)

In the third section we shall prove

**Lemma 2.** There exists an integral lattice \( S \) for which \((*)\) holds.

It is evident that these lemmas imply the theorem.

Remark. The above theorem as well as these lemmas remain true if \( L \) is a finite sum of closed curves.

2. Proof of Lemma 1. Let \( p \) be a point. We shall denote the set \([p]\) also by \( p \). The sets

\[ [q: q \in S, p + q \in G], \quad [q: q \in S, p \in G_{-q}] \]
have the same number of elements since \( p \circ G = p \circ G \). The first of them counts evidently \( f(p) \) elements. If we denote by \( g_\alpha(p) \) the characteristic function of \( G_v \), then the number of elements of the second set is \( \sum_{\alpha} g_\alpha(p) \). So we obtain
\[
(1) \quad f(p) = \sum_{\alpha} g_\alpha(p).
\]

We define \( Q = \{(x, y): 0 < x, y \leq 1\} \). Then evidently
\[
(2) \quad m \leq \int_Q f(p) \, dp \leq N.
\]

From (1) follows
\[
(3) \quad \int_Q f(p) \, dp = \sum_{\alpha} \int_Q g_\alpha(p) \, dp = \sum_{\alpha} \int_Q g_\alpha(p) \, dp = |G|
\]
(here the sums are finite since \( G \) is bounded \(^{(1)}\)). From (2) and (3) follows
\[
m \leq |G| \leq N.
\]

Let us now prove that the assumption that Lemma 1 is not true yields a contradiction. For this aim we suppose that some integer \( k \) satisfies \( m \leq k \leq M \) and \( f(p) \neq k \) for each \( p \). Thus the sets \( A = \{ p: f(p) < k \} \) and \( B = \{ p: f(p) > k \} \) which are of course both of non zero measure, cover the plane. It follows from well known results that the set \( \bigcup_{\alpha \in A} (L_\alpha \cap L_\beta) \) where the summation runs as in \((*)\) does not cut the plane, i.e. its complement \( C \) is a connected set \((*)\). Since the measure of \( \bigcup_{\alpha \in A} (L_\alpha \cap L_\beta) \) is zero we obtain that \( A \cap C \) and \( B \cap C \) are not empty. Consequently there exists a point \( r \in C \) which is common to their closures. Evidently we can find such a neighbourhood \( W \) of \( r \) which intersects at most one of the sets \( L_\alpha \) where \( \alpha \in \mathbb{S} \). Then for \( p \in W \) at most one point of the lattice \( S_\alpha \), namely \( p_\alpha \), lies on \( L \). Thus the function \( f(p) \) takes on one or two values which are consecutive integers. We arrived to a contradiction since \( W \cap A \cap C \) and \( W \cap B \cap C \) are not empty sets.

5. Proof of Lemma 2. Let us denote by \( S_\alpha \) the set obtained from \( S \) by a rotation around \( O \) through the angle \( \varphi \). For two sets \( A, B \) we shall write
\[
A \sim B
\]
if \( B \) is a translation of \( A \) given by a vector \( O \varphi \) where \( p \circ S_\alpha \). It is evident that \((*)\) holds for \( S_\alpha \) if and only if no arc \( L', L'' \subset L \) satisfy \( L' \sim L'' \).

Such angle \( \varphi \) exists since
\[(*)\quad \text{The set of all angles \( \varphi \) for which there exist such arcs \( L', L'' \subset L \) that \( L' \sim L'' \) is at most countable.}
\]

So it remains to prove \((***)\). Let us denote by \( B \) a denumerable set of arcs such that each arc \( L' \subset L \) contains an arc which belongs to \( B \). Evidently it is sufficient to prove \((***)\) under the assumption that \( L' \subset W \). Let us observe that \((**)\) follows from the assertions

I. For any two arcs \( L', L'' \subset L \) there exists only a finite number of such \( q \) that \( L' \sim L'' \).

II. For any arc \( L' \subset W \) the set of all arcs \( L'' \subset L \) for which there exist such \( q \) that \( L' \sim L'' \) holds is finite.

Proof of I. Let \( \varphi \) be the vector which defines the translation transforming \( L' \) on \( L'' \). Evidently there exist only a finite number of integral lattices \( S_\alpha \) for which \( p \circ S_\alpha \).

Proof of II. We suppose that \( L \) is a directed curve so that any two points \( u, v \) define an arc \( (u, v) \) of \( L \) with the initial point \( u \) and the endpoint \( v \). We shall write
\[
(u, v) \sim (p, v)
\]
if \( p(u, v) \) and if for the translation \( T \) defined by \( T_0 = u \) it is \( T(p, v) = (u, T(v)). \) Let us observe the following property \( P_1 \) of \( u \):
\[
(P_1) \quad \text{If } (u, v) \sim (p, v), T(p, v) = (u, T(v)) \text{ and } q \in (u, T(v)), (u, v) \sim (q, v), \text{ then } (u, v) \sim T^{-1}(q).
\]

Let us denote by \( Q \) the set of all \( p \) which satisfy \( (u, v) \sim (p, v) \). We shall say that \( p \) is a left limit point of a set \( E \) if in each arc \( (p, q) \) are points of \( E \) (different from \( p \)). Then it is
\[
(P_2) \quad \text{If } u \text{ is a left limit point of } Q \text{ then } Q \cap (u, v) \text{ is a segment.}
\]

Let us prove \( P_1 \). From \( P_1 \) follows that each point of \( Q \) is a left limit point of \( O \). Now \( O \) is a closed set and thus if some point \( q \in (u, v) \) does not belong to \( Q \) then there exists such an arc \( (a, b) \subset (u, v) \) that \( (a, b) \cap Q = \varnothing \). This is impossible since \( a \) is a left limit point of \( Q \).

If there exists a point \( q \in (u, v) \) which does not belong to the segment \( [u, v] \), then there exists such an arc \((a, b) \subset (u, v) \) that \( (a, b) \cap [u, v] = \varnothing \). Thus \( a \) is a left limit point of such points \( q \) for which the distance \( d(q) \) from the straight line \( uu \) is greater than \( d(a) \). From \( (u, v) = Q \) follows that each point \( p \in (u, v) \) which lies on the same side of the line \( uu \) as \( (a, b) \) also has the above property of \( a \). This is in contradiction with the existence of such points \( p \) which lie on the same side of \( uu \) as \( (a, b) \) and on which \( d(p) \) attains its maximum.
Let us suppose now that $\Pi$ does not hold. Then there exist an arc $L' = (p, q)$ and a sequence of arcs $(u_i, v_i)$ such that $(p, q) \sim (u_i, v_i)$. 

For each $i$ one of the vectors $pu_i$, $pv_i$ is the translation vector and thus it has the length $\sqrt{m}$ where $m$ is an integer. Since $L$ is bounded, there is only a finite number of lengths possible for $pu_i$ or $pv_i$. Therefore we may assume (restricting ourselves to a subsequence $\{(u_n, v_n)\}$) that for some $m$ all points $u_i$ or all points $v_i$ lie on a circle of radius $\sqrt{m}$.

We can assume also that there exist

\[
\lim_{i \to \infty} u_i = u \quad \text{and} \quad \lim_{i \to \infty} v_i = v.
\]

Then, for sufficiently large $i$, $(u, v) \sim (u_i, v_i) = (u_i, v)$ or $(u, v) = (u, v)$.

Since the second of these possibilities can be reduced to the first by changing the sense on $L'_i$, we can assume that for every $i$

\[
(u, v) \sim (u_i, v_i) = (u, v).
\] (4)

Since the arcs $(u_i, v_i)$ are congruent one to another by translations, each of these arcs is congruent by translation to $(u, v)$. Consequently (4) implies that $(u, v) \sim (u_i, v)$ for each $i$. Thus from (P$_3$) follows that $(u, v)$ is a segment. But then $(u_i, v_i)$ are segments of a straight line.

This is impossible since we obtained that all points $u_i$ or all points $v_i$ lie on a circle. Thus the assumption that $\Pi$ does not hold leads to a contradiction.

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