in which denumerably long atomic formulas may occur, and disjunctions and conjunctions of systems of formulas with the power of the continuum may be formed(5).

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(5) We should like to indicate here some other publications in which logical systems with infinitely long expressions are directly or indirectly involved, in fact, [1], [2], [3], and [4]. In particular, the observations in this note are related to some results in [2]. While the discussion in [2] is lacking a precisely defined logical and set-theoretical basis, it seems that the results of this discussion could be (and probably ought to be) interpreted as belonging to the theory of models for predicate logic $\mathbf{P}_\infty$ with arbitrarily long infinite expressions.

The seminar in the foundations of mathematics conducted by L. Henkin and A. Tarski at the University of California at Berkeley in the fall semester of 1956 was entirely devoted to the discussion of predicate logics with infinitely long expressions. In particular, Henkin and Tarski communicated some new results in this field (not yet published), and Mrs. Carol Karp gave a detailed report on her investigations into the syntax of such logics.

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HOMOLOGICAL RINGOIDS

BY

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1. Introduction. The algebraic study of homology theory may be said to have originated with Poincaré who associated with every compact polyhedron certain numerical invariants, the so-called *Betti numbers* and *torsion coefficients* of the polyhedron (1). Emmy Noether is credited with the observation that these numerical invariants were in fact invariants of certain finitely-generated Abelian groups, the *homology groups* of the given polyhedron.

More precisely, given any simplicial complex K triangulating a polyhedron $|K|$, one considers, for each dimension n, chains of n-simplexes of the triangulation K, such an n-chain being abstractly an element of the free Abelian group freely generated by the n-simplexes. The boundary of any (oriented) n-simplex is a well-defined $(n-1)$-chain (consisting of the suitably oriented $(n-1)$-faces of the simplex) and one obtains in this way a homomorphism ∂_n from C_n, the group of n-chains, to C_{n-1} (2). The n-cycles of K are the n-chains in the kernel of ∂_n and the n-boundaries of K are the n-chains in the image of ∂_{n+1}. The fundamental relation $\partial_{n+1}\partial_n = 0$ (homomorphisms are here written on the *right*, so $\partial_{n+1}\partial_n$ means ∂_{n+1} *followed by* ∂_n) implies that the group of n-boundaries $B_n(K)$ is a subgroup of the group of n-cycles $Z_n(K)$ and so a factor group $H_n(K) = Z_n(K)/B_n(K)$ is defined. This factor group is the *n-th homology group* of K and it may be shown that if K, L are triangulations of homeomorphic polyhedra then their homology groups are isomorphic; briefly *the homology groups are topological invariants*. Moreover, the Betti numbers and torsion coefficients of dimension n of the polyhedron $|K|$ are the rank and invariant factors of the finitely-generated Abelian group $H_n(K)$.

To-day the scope of homology theory is very broad. There are various homology theories for general spaces (*e. g.* singular theory, Čech theory); there is a dual theory of cohomology in which additional elements of alge-

(1) It is believed that Heegard pointed out to Poincaré the possibility of torsion in homology relations of cycles.

(2) We may put $C_{-1} = 0$, $\partial_0 = 0$.

braic structure make their appearance (*e. g.* cohomology ring, Steenrod powers); there are homology and cohomology groups with respect to arbitrary Abelian coefficient groups and even generalizations of these (*e. g.* local coefficients, coefficient sheaves); and there are abstract homology and cohomology theories in which the underlying objects for the theory are not topological spaces but are themselves algebraic structures (*e. g.* (non-Abelian) groups, associative algebras, Lie algebras).

It is thus necessary considerably to widen the definition of homology theory in order to include its various aspects. In [3], Cartan and Eilenberg describe a purely algebraic theory, called *homological algebra*, in which the algebraic constructions and operations employed in the various applications of homology theory are presented within a unified framework. In this paper we are concerned with the question of presenting certain fundamental notions of homological algebra.

At the first level of generality it appears that a basic concept in homology theory is that of a *differential group*; that is, an Abelian group C together with an endomorphism $\partial: C \to C$ such that $\partial^2 = 0$. If Z is the kernel of ∂ and $B = C\partial$ then $B \subset Z$ and $H(C)$, the homology group of (C, ∂), is defined as Z/B. If C is graduated as $\sum_n C_n$ and $C_n \partial \subset C_{n-1}$, then C is called a *chain complex* and $H(C)$ inherits a graduation from C. A map $\Phi: C \to C'$ of differential groups is *homomorphic* if it preserves the differential group structure; that is, if it is a group homomorphism and satisfies $\Phi\partial' = \partial\Phi$. Then Φ induces in an evident way a homomorphism $\Phi_*: H(C) \to H(C')$. The tranformation $(C, \Phi) \to (H(C), \Phi_*)$ may be called the *homology functor* H; it passes from the category (³) of differential groups and (differential group-) homomorphisms to the category of Abelian groups and homomorphism. If C, C' are chain complexes and Φ preserves graduation, then so does Φ_*.

In the classical homology theory of polyhedra we have essentially the following situation. Let (⁴) $\boldsymbol{K}$ be the category of simplicial complexes, let $\boldsymbol{P}$ be the category of polyhedra, let $\boldsymbol{C}$ be the category of chain complexes and let $\boldsymbol{A}_G$ be the category of graduated Abelian groups. Then there are functors

$$\boldsymbol{P} \overset{P}{\leftarrow} \boldsymbol{K} \overset{C}{\rightarrow} \boldsymbol{C} \overset{H}{\rightarrow} \boldsymbol{A}_G$$

where P associates with a complex its underlying polyhedron and C associates with a complex the chain complex generated by its simplexes.

(³) The notions of *category* and *functor* are to be found, *e. g.* in [4].

(⁴) For the sake of simplicity, we omit the description of the maps in the categories $\boldsymbol{K}, \boldsymbol{P}, \boldsymbol{C}, \boldsymbol{A}_G$.

Then the invariance of the homology groups is expressed by saying that if, for two objects K, L of $\boldsymbol{K}$, $P(K)$ and $P(L)$, are equivalent in $\boldsymbol{P}$, then $HC(K)$ and $HC(L)$ are equivalent in $\boldsymbol{A}_G$.

If we are not concerned with the *graduated* structure of the chain complex we may replace $\boldsymbol{C}$, $\boldsymbol{A}_G$ by $\boldsymbol{D}$, $\boldsymbol{A}$, where $\boldsymbol{D}$ is the category of differential groups and $\boldsymbol{A}$ is the category of Abelian groups; several results and concepts associated with the homology functor $\boldsymbol{C} \to \boldsymbol{A}_G$ are specializations of corresponding features of the homology functor $\boldsymbol{D} \to \boldsymbol{A}$ (*e. g.* exact homology sequences). At the next level of generality it appears undesirable and unnecessary to restrict the underlying algebraic structure of the category D to that of an Abelian group. Maclane [6] and Buchsbaum [2] have given axioms for a category, called by Maclane an *Abelian* category and by Buchsbaum an *exact* category, which are sufficient to enable the passage to homology to be effected within the category. That is to say, if $\boldsymbol{E}$ is any exact category, then the maps of ε have kernels and images and quotients of objects by subobjects may be taken; thus it is meaningful to assert that a map $\partial: E \to E$ in $\boldsymbol{E}$ satisfies $\partial^2 = 0$ and it is possible to define the homology object associated with any such pair (E, ∂).

In many aguments in present-day algebraic topology (and, more generally, in homological algebra) diagrams play an important role. A simple case of such a diagram is a square

$$\begin{array}{ccc} A & \overset{\varphi}{\rightarrow} & A' \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha'} \\ B & \overset{\psi}{\rightarrow} & B' \end{array}$$

where, for example, each of A, A', B, B' is an Abelian group, each of φ, ψ, α, α' is a homomorphism and the diagram is *commutative* in the sense that $\alpha\psi = \varphi\alpha'$. In the arguments to which we refer it is noteworthy that the fact that the objects $A, A', \ldots$ possess elements appears to play a subsidiary role, the main role being played by such facts as that the maps $\varphi, \psi, \ldots$ possess kernels and images, that the maps (homomorphisms) $A \to B$, for fixed A, B, may be "added" and that maps $A \to B, B \to C$, for fixed A, B, C, may be "multiplied". These observations have provided a strong motivation for the axioms of an exact category. It may be said, roughly speaking, that Maclane and Buchsbaum proceed by "throwing away" the elements, retaining only the objects and the maps and insisting on certain important properties exhibited by the category $\boldsymbol{A}$ of Abelian groups.

In our approach we are guided by similar considerations to those of Maclane and Buchsbaum; again speaking approximately, we attempt in our treatment to throw away both the elements and the objects, retaining only the maps. More precisely, the objects remain as suppressed indices for the maps but do not appear in the notation(5); thus we do not at this stage attempt to broaden the scope of homological algebra beyond exact categories, but change the emphasis by notational and conceptual simplification.

The algebraic object which we study is a *ringoid* in the sense of M. G. Barratt [1]. This is a ring except that the ring operations are not always defined. Then the maps of the category A form a ringoid and we introduce axioms, derived from those of Buchsbaum (Axioms I-IV of [2]), which imitate certain essential features of the ringoid of maps of A. However our axioms are expressed in terms of the classical notions of ring theory. It appears to us that, when presented in this way within the cadre of ring theory, many of the notions of homological algebra take on an appearance familiar to the algebraist and the arguments are, as we have said, in some places conceptually simplified. We stress particularly that the strict duality present in an exact category, which is an important feature of Buchsbaum's treatment, is also given a central position in our axiomatization.

An application of homological ringoids to the classical theory of finitely generated chain complexes (not necessarily free Abelian) is to be found in [5]. Details of the general theory will be published later.

2. Homological ringoids. Before presenting the axioms of a homological ringoid in an abstract form, we return to the concrete case of the category A of Abelian groups. For maps in this category (*i. e.* homomorphisms) addition and multiplication are sometimes defined if α_1 and α_2 are maps from A into B, then so is $\alpha_1+\alpha_2$; and if α maps A into B and β maps B into C, then $\alpha\beta$ is the resulting map from A into C, consisting of α followed by β. The set of maps then forms a *ringoid* in the sense that the usual ring axioms, namely

$$(2.1)\qquad \begin{cases} \alpha_1+\alpha_2=\alpha_2+\alpha_1, \quad (\alpha_1+\alpha_2)+\alpha_3=\alpha_1+(\alpha_2+\alpha_3),\\ (\alpha\beta)\gamma=\alpha(\beta\gamma),\\ (\alpha_1+\alpha_2)\beta=\alpha_1\beta+\alpha_2\beta, \quad \beta(\alpha_1+\alpha_2)=\beta\alpha_2+\beta\alpha_2 \end{cases}$$

are satisfied whenever the compositions of maps occurring in them are defined. Moreover, for fixed A, B, the set of maps $A\to B$ forms an additive Abelian group. The conditions under which maps are composable for multiplication are just those satisfied by the maps of any category. Similarly, we postulate the existence of identities in a ringoid, that is of maps ι such that

$$(2.2)\qquad \xi\iota=\xi, \quad \iota\eta=\eta,$$

whenever the products $\xi\iota$ and $\iota\eta$ are defined. When there is no danger of confusion, identity maps may indiscriminately be denoted by 1. The *zero map* from A to B in A is the map which sends every element of A into the zero element of B. No confusion arises if the zero map between any two groups is simply denoted by 0, since the terminals of each particular zero map can easily be deduced from the context.

Apart from the general laws governing composition and the existence of identities, the collection of maps of Abelian groups possesses certain typical features which correspond to the fact that every map (*i. e.* homomorphism) has a kernel and a cokernel. In any ringoid, the set of elements ξ which annihilate a given element α on the left, so that $\xi\alpha=0$, forms a left ideal $\boldsymbol{L}^0\alpha$ which we call the *left annihilator* of α. It turns out that in the present case these annihilators are always principal ideals, that is, there exists a map μ such that $\mu\alpha=0$, and $\xi=\varphi\mu$ whenever $\xi\alpha=0$. We then write $\boldsymbol{L}^0\alpha=\boldsymbol{L}(\mu)$. Moreover, μ is a monomorphism or, as we shall say, is *left-regular*; that is, $\xi\mu=0$ implies that $\xi=0$. In fact, if α maps A into B, μ is the map which embeds the kernel of α in A.

Similarly, in any ringoid, the right annihilator of an element α is a right ideal $\boldsymbol{R}^0\alpha$, but in this special ringoid it is always a principal right ideal; thus

$$\boldsymbol{R}^0\alpha=\boldsymbol{R}(\varepsilon)$$

where ε is an epimorphism or, *right-regular*, which means that $\varepsilon\xi=0$ implies that $\xi=0$. If B_0 is the exact image of A under α, then ε is the map which projects B onto B/B_0.

The concept of a homological ringoid is derived by abstraction from the collection of maps between Abelian groups. Since we wish to emphasize the algebraical structure of such a ringoid, we shall henceforth speak of elements rather than maps. Nevertheless it will still be convenient at times to employ diagrams such as

$$\xrightarrow{\alpha}\cdot\xrightarrow{\beta}$$

to indicate that the product $\alpha\beta$ is defined.

We say then that a set $\boldsymbol{H}$ of elements $\alpha,\beta,\ldots$ forms a *homological ringoid* if it is a ringoid and if, in addition, the following axioms are satisfied:

(5) Abelian group theory customarily employs 3 alphabets: one (capital Roman) for groups, one (small Roman) for group elements, and one (small Greek) for homomorphisms, Maclane and Buchsbaum dispense with small Roman letters and use 2 alphabets. We dispense with large Roman letters, too!

AXIOM I. (i) *For every left-regular μ there exists a right-regular ε such that $\boldsymbol{R}^0\mu = \boldsymbol{R}(\varepsilon)$.*

(ii) *For every right-regular ε, there exists a left-regular μ such that $\boldsymbol{L}^0\varepsilon = \boldsymbol{L}(\mu)$.*

AXIOM II. *For every right-regular ε, there exists a left-regular μ such that $\boldsymbol{R}(\varepsilon) = \boldsymbol{R}^0\mu$.*

(ii) *For every left-regular μ, there exists a right-regular ε such that $\boldsymbol{L}(\mu) = \boldsymbol{L}^0\varepsilon$.*

AXIOM III. *Every α is expressible in the form $\alpha = \varepsilon\mu$, where ε is right-regular and μ is left-regular.*

Let us first draw attention to some purely algebraical consequences of these axioms. Axiom I ensures that the annihilator of a left- or a right-regular map is always a principal ideal. But by Axiom III this statement is seen to hold for any α. For example, consider the right annihilator of α. This consists of all ξ such that $\alpha\xi = \varepsilon\mu\xi = 0$. Since ε is right-regular, this is equivalent to $\mu\xi = 0$; thus $\boldsymbol{R}^0\alpha = \boldsymbol{R}^0\mu = \boldsymbol{R}(\varepsilon')$ where ε' is a suitable right-regular element.

By a combination of Axioms I and II it can be shown ([5], Proposition 2.1) that *the relations $\boldsymbol{R}^0\mu = \boldsymbol{R}(\varepsilon)$ and $\boldsymbol{L}^0\varepsilon = \boldsymbol{L}(\mu)$ imply each other where μ and ε are left and right-regular respectively.* We then call μ and ε *mutual annihilators* and write

$$\mu\|\varepsilon. \tag{2.3}$$

Thus this symbol expresses the following facts: (i) $\mu\varepsilon = 0$, (ii) $\mu\xi = 0$ implies that $\xi = \varepsilon\varkappa$ and (iii) $\xi\varepsilon = 0$ implies that $\xi = \lambda\mu$. Of course, to verify (2.3), it is sufficient to verify (i) and (ii) or (i) and (iii).

The factorization postulated in Axiom III is not unique, but if $\alpha = \varepsilon\mu = \varepsilon'\mu'$, there exists a *unit* Θ, that is an element which is both left and right-regular, such that $\varepsilon' = \varepsilon\Theta$, $\mu = \Theta\mu'$.

We should remark that in order to avoid exceptional cases we have assumed that among the zeros of H there are some, say ε_0, that are right-regular and therefore admit only zeros as right factors. Similarly, there are left-regular zeros μ_0. These two types of zeros correspond in $\boldsymbol{A}$ to the zero maps $A \to 0$ and $0 \to B$ respectively.

The concept of *exactness* is formulated as follows: let $\alpha = \varepsilon_\alpha\mu_\alpha$, $\beta = \varepsilon_\beta\mu_\beta$ be two elements, each factorized in accordance with Axiom III. We then call the pair α, β *exact*, and *we write $\alpha|\beta$ if $\mu_\alpha\|\varepsilon_\beta$*. Note that the exactness of α, β does not depend on the factors μ_β or ε_α.

As an example of a type of argument which frequently occurs in this work, we consider the following proposition, to which reference will be made in the next section:

PROPOSITION 2.4. *If $\lambda\|\eta$ and $\lambda|\eta\varrho$, then ϱ is left-regular.*

Let $\eta\varrho = \varepsilon\mu$ be the factorization of $\eta\varrho$. Then $\lambda|\eta\varrho$ means that $\lambda\|\varepsilon$. But we are given that $\lambda\|\eta$. It follows that $\varepsilon = \eta\theta$ where θ is a unit. We now have $\eta\varrho = \eta\theta\mu$. Since η is right-regular, we may deduce that $\varrho = \theta\mu$, whence it is evident that ϱ is left-regular.

One of the commonest situations in homological algebra is a *commutative square*, that is a set of 4 elements $\alpha, \beta, \varrho, \sigma$ satisfying $\alpha\sigma = \varrho\beta$, as is illustrated by the diagram:

$$\begin{array}{ccc} \cdot & \xrightarrow{\alpha} & \cdot \\ {\scriptstyle\varrho}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \cdot & \xrightarrow[\beta]{} & \cdot \end{array} \tag{2.5}$$

Now let $\alpha = \varepsilon\mu$, $\beta = \eta\nu$ be the factorizations of α and β. We shall show that (2.5) *splits as follows*:

$$\begin{array}{ccccc} \cdot & \xrightarrow{\varepsilon} & \cdot & \xrightarrow{\mu} & \cdot \\ \downarrow{\scriptstyle\varrho} & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\sigma} \\ \cdot & \xrightarrow[\eta]{} & \cdot & \xrightarrow[\nu]{} & \cdot \end{array}$$

that is, there exists a unique element τ such that $\varepsilon\tau = \varrho\eta$, $\mu\sigma = \tau\nu$

Proof. Let $\mu'\|\varepsilon$. On multiplying the given equation

$$\varepsilon\mu\sigma = \varrho\eta\nu \tag{2.6}$$

on the left by μ' we obtain that $0 = \mu'\varrho\eta\nu$. Since ν is left-regular, it follows that $\mu'\varrho\eta = 0$. Now the relation $\mu'\|\varepsilon$ implies that there exists τ such that $\varrho\eta = \tau\varepsilon$; and τ is unique, because ε is right-regular. Substituting for $\varrho\eta$ in (2.6) we find that $\varepsilon\mu\sigma = \varepsilon\tau\nu$ whence, by the right-regularity of ε, $\mu\sigma = \tau\nu$, which completes the proof.

An extension of this argument leads to the more complete commutative diagram

$$\begin{array}{ccccccccc} \cdot & \xrightarrow{\mu'} & \cdot & \xrightarrow{\varepsilon} & \cdot & \xrightarrow{\mu} & \cdot & \xrightarrow{\varepsilon'} & \cdot \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varrho} & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\varphi} \\ \cdot & \xrightarrow[\nu']{} & \cdot & \xrightarrow[\eta]{} & \cdot & \xrightarrow[\nu]{} & \cdot & \xrightarrow[\eta']{} & \cdot \end{array}$$

where $\mu\|\varepsilon'$, $\nu'\|\eta$ and $\nu\|\eta'$, and where φ and ψ (and τ) are determined up to units by (2.5).

3. Applications. In this section we indicate by simple examples the techniques which may be employed to generalize to homological ringoids classical theorems of Abelian group theory and homological algebra.

Consider the classical isomorphism theorem: if X, Y, Z are Abelian groups with $Z \subset Y \subseteq X$, then $X/Y \cong X/Z/Y/Z$. We first elaborate this theorem, putting greater emphasis on the homomorphisms involved.

Let

$$0 \to Z \xrightarrow{\lambda} Y \xrightarrow{\eta} R \to 0,$$
$$0 \to Z \xrightarrow{\mu} X \xrightarrow{\varepsilon'} Q \to 0,$$
$$0 \to Y \xrightarrow{\mu'} X \xrightarrow{\varepsilon} P \to 0$$

be exact sequences such that $\mu = \lambda\mu'$.

THEOREM 3.1. (Isomorphism Theorem). *There is an exact sequence*

$$0 \to R \xrightarrow{\varrho} Q \xrightarrow{\sigma} P \to 0$$

such that the diagram

$$(3.2)\qquad \begin{array}{ccccc} Z & \xrightarrow{\lambda} & Y & \xrightarrow{\eta} & R \\ & {\scriptstyle\mu}\searrow & \downarrow{\scriptstyle\mu'} & & \downarrow{\scriptstyle\varrho} \\ & & X & \xrightarrow{\varepsilon'} & Q \\ & & & {\scriptstyle\varepsilon}\searrow & \downarrow{\scriptstyle\sigma} \\ & & & & P \end{array}$$

is commutative; moreover, ϱ and σ are uniquely determined.

We first remark that 3.1 constitutes a precise statement of the isomorphism theorem: we may write $R = Y/Z$, $Q = X/Z$, $P = X/Y$, and we prove that σ induces an isomorphism $Q/R \cong P$.

We prove the theorem without any appeal to the structure of the Abelian groups themselves, merely using the homological-ringoid structure of the homomorphisms. Thus the problem is this: given elements $\lambda, \mu, \mu', \eta, \varepsilon, \varepsilon' \in \boldsymbol{H}$ with $\lambda\|\eta, \mu\|\varepsilon', \mu'\|\varepsilon$ and $\lambda\mu' = \mu$, to establish the existence and uniqueness of elements ϱ, σ with $\eta\varrho = \mu'\varepsilon'$, $\varepsilon'\sigma = \varepsilon$ and $\varrho\|\sigma$. Thus we start with a diagram

$$(3.3)\qquad \begin{array}{ccccc} \cdot & \xrightarrow{\lambda} & \cdot & \xrightarrow{\eta} & \cdot \\ & {\scriptstyle\mu}\searrow & \downarrow{\scriptstyle\mu'} & & \vdots{\scriptstyle\varrho} \\ & & \cdot & \xrightarrow{\varepsilon'} & \cdot \\ & & & {\scriptstyle\varepsilon}\searrow & \vdots{\scriptstyle\sigma} \\ & & & & \cdot \end{array}$$

The existence of ϱ follows from the relation $\lambda(\mu'\varepsilon') = \mu\varepsilon' = 0$, since $\lambda\|\eta$; the uniqueness of ϱ follows from the fact that η is right-regular. The existence of σ follows from the relation $\mu\varepsilon = \lambda\mu'\varepsilon = 0$, since $\mu\|\varepsilon'$; the uniqueness of σ follows from the fact that ε' is right-regular. Thus we have the complete diagram

$$(3.4)\qquad \begin{array}{ccccc} \cdot & \xrightarrow{\lambda} & \cdot & \xrightarrow{\eta} & \cdot \\ & {\scriptstyle\mu}\searrow & \downarrow{\scriptstyle\mu'} & & \downarrow{\scriptstyle\varrho} \\ & & \cdot & \xrightarrow{\varepsilon'} & \cdot \\ & & & {\scriptstyle\varepsilon}\searrow & \downarrow{\scriptstyle\sigma} \end{array}$$

and it remains to prove that $\varrho\|\sigma$. First we prove that ϱ is left-regular; as observed in 2.4 it is sufficient to show that $\lambda|\eta\varrho$ or $\lambda|\mu'\varepsilon'$. Suppose then that $\xi\mu'\varepsilon' = 0$; since $\mu\|\varepsilon'$, we have $\xi\mu' = \varkappa\mu = \varkappa\lambda\mu'$. But μ' is left-regular so $\xi = \varkappa\lambda$ and $\lambda|\eta\varrho$. Thus ϱ is left-regular. From the relation $\varepsilon'\sigma = \varepsilon$ it follows that σ is right-regular, since ε is right-regular. Next $\eta\varrho\sigma = \mu'\varepsilon'\sigma = \mu'\varepsilon = 0$; but η is right-regular so $\varrho\sigma = 0$. Finally let $\varrho\xi = 0$; then $\mu'\varepsilon'\xi = 0$, whence $\varepsilon'\xi = \varepsilon\varkappa = \varepsilon'\sigma\varkappa$. But ε' is right-regular, so $\xi = \sigma\varkappa$ and the proof is complete.

We may immediately pass to the dual theorem; this asserts that, given elements $\varrho, \mu, \mu', \sigma, \varepsilon, \varepsilon' \in \boldsymbol{H}$ with $\varrho\|\sigma$, $\mu\|\varepsilon'$, $\mu'\|\varepsilon$ and $\varepsilon'\sigma = \varepsilon$, there are unique elements λ, η with $\eta\varrho = \mu'\varepsilon'$, $\lambda\mu' = \mu$, and $\lambda\|\eta$. Translating back into the category of Abelian groups we find the theorem that *if $\sigma: Q \to P$, $\varepsilon': X \to Q$, $\varepsilon: X \to P$ are epimorphisms with kernels R, Z, Y respectively and if $\varepsilon'\sigma = \varepsilon$, then ε' induces an epimorphism $Y \to R$ with kernel Z.* This theorem is not, of course, deep; we mention it to emphasize that by proving 3.1 in the ringoid $\boldsymbol{H}$ we have obtained a proof of the dual theorem.

The diagram (3.4) is particularly relevant to homology theory. *A chain complex* in $\boldsymbol{H}$ is a sequence of elements $\{\partial_n\}$ such that (6) $\partial_{n+1}\partial_n = 0$. Decompose ∂_n as $\varepsilon_n\mu_{n-1}$. Then ε_n corresponds to the (boundary) map of n-chains on $(n-1)$-boundaries and μ_n corresponds to the embedding of n-boundaries as n-chains. The relation $\partial_{n+1}\partial_n = 0$ is equivalent to $\mu_n\varepsilon_n = 0$. Let $\mu'_n\|\varepsilon_n$, $\mu_n\|\varepsilon'_n$. Then μ'_n embeds n-cycles as n-chains and $\mu_n = \lambda_n\mu'_n$ where λ_n embeds n-boundaries as n-cycles. If $\lambda_n\|\eta_n$, then η_n associates with each n-cycle its homology class. Let ϱ_n, σ_n be defined as in 3.1. The diagram

$$(3.5)\qquad \begin{array}{cccccc} {\scriptstyle\varepsilon_{n+1}}\searrow & & & & & \\ & \cdot & \xrightarrow{\lambda_n} & \cdot & \xrightarrow{\eta_n} & \cdot \\ & & {\scriptstyle\mu_n}\searrow & \downarrow{\scriptstyle\mu'_n} & & \downarrow{\scriptstyle\varrho_n} \\ & & & \cdot & \xrightarrow{\varepsilon'_n} & \cdot \\ & & & & {\scriptstyle\varepsilon_n}\searrow & \downarrow{\scriptstyle\sigma_n} \\ & & & & & \cdot \\ & & & & & {\scriptstyle\mu_{n-1}}\searrow \end{array}$$

is called the *n-th homology diagram* of the chain-complex. The duality to which we have drawn attention shows that we should obtain the same diagram (up to multiplication by units) by proceeding from the relation $\mu_n\|\varepsilon'_n$ (instead of $\mu'_n\|\varepsilon_n$). If $\{\partial_n\}, \{\bar{\partial}_n\}$ are two chain complexes a *chain map* $\{\Phi_n\}: \{\partial_n\} \to \{\bar{\partial}_n\}$ is a sequence of elements satisfying $\Phi_{n+1}\bar{\partial}_{n+1} = \partial_{n+1}\Phi_n$.

(6) We may allow $-\infty < n < +\infty$; usually $\partial_n = 0$ for $n \leqslant 0$.

Then $\{\Phi_n\}$ induces a map (in an evident sense) of the homology diagram of $\{\partial_n\}$ to that of $\{\bar{\partial}_n\}$.

As a second example we prove a "3-lemma", so-called by analogy with the well-known „5-lemma" of homological algebra. Indeed it is easy to prove the 5-lemma, in its strong form, from the 3-lemma and the splitting procedure described at the end of section 2.

THEOREM 3.6. *Given the commutative diagram*

$$\begin{array}{ccccc} \cdot & \xrightarrow{\mu} & \cdot & \xrightarrow{\varepsilon} & \cdot \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ \cdot & \xrightarrow[\nu]{} & \cdot & \xrightarrow[\eta]{} & \cdot \end{array}$$

with $\mu\|\varepsilon$, $\nu\|\eta$, *then*

(a) *if* φ *is right-regular,* ψ *is right-regular,*

(a′) *if* φ *is left-regular,* θ *is left-regular;*

(b) *if* θ *is right-regular and* φ *is left-regular, then* ψ *is left-regular,*

(b′) *if* ψ *is left-regular and* φ *is right-regular, then* θ *is right-regular;*

(c) *if* θ *and* ψ *are right-regular, so is* φ,

(c′) *if* θ *and* ψ *are left-regular, so is* φ.

We remark that (a′), (b′), (c′) are dual to (a), (b), (c) and thus do not need separate proof. We prove (a) by remarking that $\varepsilon\psi = \varphi\eta$, which is right-regular since φ and η are right-regular; thus ψ is right-regular. To prove (b) we remark that θ is a unit by (a′); the assertion is thus equivalent to the assertion, following (3.4), that ϱ is left-regular. To prove (c) let $\varphi\xi = 0$; then $\theta\nu\xi = \mu\varphi\xi = 0$, so $\nu\xi = 0$, θ being right-regular. Thus $\xi = \eta\varkappa$, so $0 = \varphi\eta\varkappa$; but $\varphi\eta = \varepsilon\psi$ is right-regular, so $\varkappa = 0$, $\xi = 0$, and φ is right-regular.

COROLLARY 3.7. *If any two of* θ, φ, ψ *are units, so is the third.*

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SUR LES ENSEMBLES DENSES DE PUISSANCE MINIMUM DANS LES GROUPES TOPOLOGIQUES

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Dans les espaces non métrisables, même s'ils sont compacts, les deux axiomes de dénombrabilité, l'existence d'un ensemble dénombrable dense ou bien la non-existence d'un ensemble indénombrable isolé, ou encore, par exemple, la non-existence d'une famille indénombrable de voisinages disjoints présentent des propriétés essentiellement différentes. Néanmoins, quelques implications mutuelles non triviales ont lieu dans le cas d'un groupe topologique soumis, s'il y a besoin, à des conditions supplémentaires. Ainsi par exemple, selon le théorème connu de Kakutani, le premier axiome de dénombrabilité entraîne dans tout groupe topologique l'existence d'une métrique invariante; dans un groupe compact en résultent, par conséquent, toutes les autres propriétés mentionnées ci-dessus. De plus, on sait que, si l'on admet l'hypothèse du continu, un groupe compact abélien de puissance $2^{\aleph_0}$ est pourvu d'un système fondamental dénombrable, et A. Hulanicki a démontré récemment [3] que c'est encore le cas des groupes localement compacts non-abéliens.

Dans cette note, nous nous proposons d'établir les conditions qui, imposées à un groupe topologique, assurent l'existence d'un sous-ensemble dénombrable partout dense et dont le nombre cardinal est aussi petit que possible. Parallèlement, nous allons considérer l'existence de „grands" sous-groupes isolés.

LEMME 1 ([7], p. 138) (1). *Si un sous-groupe invariant fermé H d'un groupe topologique G et le groupe quotient G/H contiennent chacun un ensemble dense dont la puissance ne surpasse pas $\mathfrak{m} \geqslant \aleph_0$, il en est de même pour le groupe G.*

LEMME 2. *Un groupe topologique contenant un sous-groupe isolé de puissance $\mathfrak{m}$ contient un système de puissance $\mathfrak{m}$ de voisinages disjoints* (2).

(1) Ce lemme y est démontré pour $\mathfrak{m} = \aleph_0$, mais la démonstration est la même quel que soit $\mathfrak{m} \geqslant \aleph_0$.

(2) Cette implication est fausse pour un espace topologique quelconque, le mot „sous-groupe" étant remplacé par „sous-ensemble"; cf., par exemple, [5], p. 133.