THEOREM 2 (COMPLETENESS THEOREM FOR ARBITRARY CALCULI). *If a is any cardinal number, then an a-wff is an a-theorem in the strict sense if and only if it is an a-tautology.*

It is well-known that in the case $a = \omega$ there is an even stronger completeness result: any ω, ω-consistent class of ω-wffs has a substitution giving all formulas in the class the value T. For all $a > \omega$ it can be shown that such a stronger result fails unless possibly a is a strongly inaccessible cardinal number. Whether the stronger theorem is true in the inaccessible case is an open question (P 250) seemingly involving fundamental set-theoretical problems[8]. However, certain stronger results are possible: for example, it should be clear from the proof of Theorem 1 that every at most denumerable and ω,ω_1-consistent class of ω_1-wffs has a substitution giving all formulas in the class the value T.

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(8) This problem is directly related to those problems about inaccessible numbers formulated at the end of Erdös-Tarski [2].

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REMARKS ON PREDICATE LOGIC WITH INFINITELY LONG EXPRESSIONS

BY

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As extensions of ordinary first order predicate logic P_0 various systems of predicate logic with infinitely long expressions can be considered[1].

To fix the ideas we restrict ourselves to the discussion of predicate logic P_1 with denumerably long expressions. Atomic formulas in P_1 are expressions like

$$\varphi(v_0 \ldots v_{n-1})$$

consisting of a predicate φ and a finite sequence of variables $\langle v_0, \ldots, v_{n-1}\rangle$. The set of all predicates is assumed to be at most denumerable (though this restriction is not essential) and to contain the binary identity predicate $=$; instead of $=(v_0, v_1)$ we write $(v_0 = v_1)$. Compound well-formed formulas are obtained from simpler ones by means of the following operations: (i) the formation of the negation $\sim F$ of a formula F; (ii) the formation of the implication $[F_0 \to F_1]$ of two formulas F_0 and F_1; (iii) the formation of the disjunction

$$\bigvee [F_0 \ldots F_\xi \ldots]$$

and the conjunction

$$\bigwedge [F_0 \ldots F_\xi \ldots]$$

of a finite or denumerable sequence of formulas $\langle F_0, \ldots, F_\xi, \ldots\rangle$; (iv) the universal quantification

$$(\forall v_0 \ldots v_\xi \ldots) F$$

(1) This note contains the text of the remarks made by the author at the Summer Institute of Symbolic Logic in 1957 at Cornell University; it first appeared (under the same title, though in a more concise form) in *Summaries of talks presented at the Summer Institute of Symbolic Logic in 1957 at Cornell University*, vol. 1, p. 160-163 (mimeographed). The results of this note were obtained and the note was prepared for publication while the author was working on a research project in the foundations of mathematics sponsored by the National Science Foundation and carried through in the University of California, Berkeley.

and the existential quantification

$$(\exists v_0 \dots v_\xi \dots) F$$

of a formula F over a finite or denumerable sequence of variables $\langle v_0, \dots, v_\xi, \dots\rangle$. The notion of a free (or bound) occurrence of a variable in a formula is defined in the usual way. A formula without free occurrences of variables is called a sentence. A universal sentence is a sentence of the form

$$(\forall v_0 \dots v_\xi \dots) F$$

where F is a formula without quantifiers.

In this note we shall not attempt to define for P_1 such fundamental syntactical notions as provability or derivability; we shall be concerned exclusively with some semantical and, specifically, model-theoretical problems.

There is no difficulty in extending basic semantical notions to the logic P_1; in particular, it is clear under what conditions a relational system

$$\mathfrak{A} = \langle A, R_0, \dots, R_\xi, \dots\rangle$$

is regarded as a *model* of a sentence S in P_1 or of a set Σ of such sentences. Each of the relational systems $\mathfrak{A}$ involved here is formed by a non-empty set A and by a finite or denumerable sequence of finitary relations $\langle R_0, \dots, R_\xi, \dots\rangle$ among elements of this set. A class $\boldsymbol{K}$ of relational systems is called an *arithmetical* P-*class* or, simply, a P-*class* if it coincides with the class of all models of some set of sentences in the logic P; it is called a *universal* P-*class* if it coincides with the class of all models of some set of universal sentences in P. For P we can take here P_0, P_1, or any other logical system which may be mentioned in our further discussion(²).

Various notions applying to arbitrary relational systems (such as similar systems, at most denumerable system, isomorphic image, subsystem, and extension of a system, union of a class of systems) are assumed to be known. A non-empty class $\boldsymbol{L}$ of relational systems is called *directed (denumerably directed)* if, for every finite (at most denumerable) subclass $\boldsymbol{M}$ of $\boldsymbol{L}$, all systems in $\boldsymbol{M}$ have a common extension which belongs to $\boldsymbol{L}$. See here [6], part I, p. 573 ff. (A bibliography is given at the end of this note).

Several known results and observations in the theory of models can be extended in an appropriate form from the logic P_0 to the logic P_1. As an example we state the following

(²) The notion of a P_1-class (or a universal P_1-class) can also be defined in purely mathematical terms, without involving the logic P_1 itself or any other logical formalism. Compare [9] for an analogous definition of P_0 classes (arithmetical classes).

THEOREM 1. *For a class $\boldsymbol{K}$ of (similar) relational systems to be a universal P_1-class it is necessary and sufficient that $\boldsymbol{K}$ satisfy the following three conditions:*

(i) *if a system belongs to $\boldsymbol{K}$, then all its isomorphic images belong to $\boldsymbol{K}$;*

(ii) *if a system belongs to $\boldsymbol{K}$, then all its subsystems belong to $\boldsymbol{K}$;*

(iii) *if a denumerably directed class of systems is included in $\boldsymbol{K}$, then the union of this class belongs to $\boldsymbol{K}$.*

Condition (iii) *can be equivalenty replaced by:*

(iii′) *if every at most denumerable subsystem of a system belongs to $\boldsymbol{K}$, then the system itself belongs to $\boldsymbol{K}$.*

The proof of this theorem follows the lines of the proof of an analogous theorem for the logic P_0, in fact, of Theorem 1.2 in [6]. The argument can be conveniently based upon the following simple

LEMMA. *For every at most denumerable relational system $\mathfrak{A}$ a universal sentence S in P_1 can be constructed such that a system $\mathfrak{B}$ is a model of S if and only if no subsystem of $\mathfrak{B}$ is an isomorphic image of $\mathfrak{A}$.*

In connection with this lemma compare Theorem 1.1 in [6] and its proof.

For illustration consider the class $\boldsymbol{W}$ of all well-ordered systems $\langle A, R\rangle$ (such as the system $\langle \omega, \leqslant\rangle$ formed by the set ω of all natural numbers and the ordinary relation $\leqslant$). It has been shown that $\boldsymbol{W}$ is not a P_0-class; cf. [7], part II, p. 301, or else [8], p. 382. On the other hand it is well known that the class $\boldsymbol{K} = \boldsymbol{W}$ satisfies conditions (i) and (ii) of Theorem 1, and it is easily seen that it also satisfies condition (iii) (³). Consequently, by Theorem 1, $\boldsymbol{W}$ must be a universal P_1-class; and, in fact, $\boldsymbol{W}$ proves to coincide with the class of models of the following three universal sentences (in which the binary predicates φ and $=$ occur as the only non-logical constants):

(1) $$(\forall v_0 v_1)[\varphi(v_0 v_1) \to [\varphi(v_1 v_0) \to (v_0 = v_1)]],$$

(2) $$(\forall v_0 v_1 v_2)[\varphi(v_0 v_1) \to [\varphi(v_1 v_2) \to \varphi(v_0 v_2)]],$$

(3) $$(\forall v_0 \dots v_n \dots) \bigvee [\varphi(v_0 v_1) \dots \varphi(v_n v_{n+1}) \dots].$$

Here $\langle v_0, \dots, v_n, \dots\rangle$ is an arbitrary simple infinite sequence (of type ω) of distinct variables. The proof that $\boldsymbol{W}$ satisfies condition (iii) of Theorem 1 and coincides with the class of all models of sentences (1)-(3) is based upon the axiom of choice.

(³) The fact that $\boldsymbol{W}$ satisfies (iii) has recently been noticed by William Hanf and Bjarni Jónsson, who also pointed out that the union of an arbitrary directed class of well-ordered systems is not, in general, a well-ordered system.

With sentence (3) appropriately changed, all remarks in the preceding paragraph extend to the class $\boldsymbol{S}$ of scattered ordered systems.

We shall state still another, related result, which applies, however, not to arbitrary relational systems, but exclusively to algebraic systems (algebras). The logical basis is provided by the logic P_1' which differs from P_1 in that it contains no predicates with the exception of the identity symbol, but contains finitary operation symbols instead. The set of operations in each of the algebras involved and the set of operation symbols in P_1' are assumed to be at most denumerable.

THEOREM 2. *For a class $\boldsymbol{K}$ of (similar) algebras to coincide with the class of models of a set Σ of universal sentences in P_1', each of which contains only finitely many distinct variables, it is necessary and sufficient that $\boldsymbol{K}$ satisfy the following three conditions:*

(i) *if an algebra belongs to $\boldsymbol{K}$, then all its isomorphic images belong to $\boldsymbol{K}$;*

(ii) *if an algebra belongs to $\boldsymbol{K}$, then all its subalgebras belong to $\boldsymbol{K}$;*

(iii) *if a directed class of algebras is included in $\boldsymbol{K}$, then the union of this class belongs to $\boldsymbol{K}$.*

Condition (iii) *can be equivalently replaced by*:

(iii') *if every finitely generated subalgebra of an algebra belongs to $\boldsymbol{K}$, then the algebra itself belongs to $\boldsymbol{K}$.*

The proof is entirely analogous to that of Theorem 1 (or Theorem 1.2 in [6]) and is based upon the following

LEMMA. *For every finitely generated algebra $\mathfrak{A}$ a universal sentence S in P_1', with finitely many distinct variables, can be constructed such that an algebra $\mathfrak{B}$ is a model of S if and only if no subalgebra of $\mathfrak{B}$ is an isomorphic image of $\mathfrak{A}$.*

For illustration consider the class $\boldsymbol{T}$ of all torsion groups, *i. e.*, of all groups without elements of infinite order. It is known that $\boldsymbol{T}$ is not a P_0'-class (where P_0' is the logical system related to P_0 in exactly the same way in which P_1' is related to P_1); cf. [5], Corollary 6.10, p. 269. It is easily seen, however, that $\boldsymbol{T}$ satisfies conditions (i)-(iii) of Theorem 2 and hence coincides with the class of models of a set Σ of universal sentences in P_1', each of which contains only finitely many distinct variables. In fact, we can take for Σ the set of the following three sentences (in which the binary operation symbol $\circ$ and the identity symbol occur as the only non-logical constants):

(1) $$(\forall v_0 v_1 v_2)\big[\big(v_0\circ(v_1\circ v_2)\big) = \big((v_0\circ v_1)\circ v_2\big)\big],$$

(2) $$(\forall v_0 v_1)\vee\big[\big(v_0 = (v_1^1\circ v_0)\big)\ldots\big(v_0 = (v_1^{n+1}\circ v_0)\big)\ldots\big],$$

(3) $$(\forall v_0 v_1)\vee\big[\big(v_0 = (v_0\circ v_1^1)\big)\ldots\big(v_0 = (v_0\circ v_1^{n+1})\big)\ldots\big].$$

Here the meaning of the symbol v_1^n is determined recursively as follows: v_1^1 coincides with v_1, v_1^{n+1} coincides with $(v_1^n\circ v_1)$ for every positive natural number n [4].

There are, of course, some essential model-theoretical differences between logics P_0 and P_1. For instance, Theorems 1.6 and 1.7 in [6] do not extend to the logic P_1. More specifically, a class $\boldsymbol{K}$ of relational systems can be exhibited which is a P_1-class and satisfies condition (ii) of Theorem 1, but which is not a universal P_1-class. Such is, e. g., the class of all relational systems $\mathfrak{A} = \langle A, R\rangle$ where A in an arbitrary at most denumerable set and R is the binary universal relation in A, i. e., the relation holding between any two elements of A. $\boldsymbol{K}$ is a P_1-class since it coincides with the class of all models of the following two sentences:

(1) $$(\exists v_1\ldots v_{n+1}\ldots)(\forall v_0)\vee[(v_0 = v_1)\ldots(v_0 = v_{n+1})\ldots],$$

(2) $$(\forall v_0 v_1)\varphi(v_0 v_1).$$

Obviously, $\boldsymbol{K}$ satisfies condition (ii) of Theorem 1. However, $\boldsymbol{K}$ does not satisfy condition (iii') of the same theorem and hence is not a universal P_1-class. Thus we see that, in opposition to what is true for P_0-classes by virtue of Theorem 1.7 in [6], the P_1-classes which satisfy condition (ii) of Theorem 1 do not coincide with the universal P_1-classes. The problem of finding a purely metamathematical (model-theoretical) characterization for P_1-classes satisfying condition (ii) of Theorem 1 is open (P 251).

Observations entirely analogous to those made in this note apply to logics P_α for an arbitrary ordinal α. By P_α we understand predicate logic constructed analogously to P_0 and P_1 but in which arbitrary sequences of variables and formulas of any type smaller than the initial ordinal ω_α are used.

While in this note we have concerned ourselves with logics in which all atomic formulas are finite, logical systems with infinitely long atomic formulas can be studied as well. In particular, in order to extend theorems 1 and 2 to relational systems with infinitary relations and to algebras with infinitary operations of denumerable rank, we need a logical system

[4] This opportunity is taken to correct an error in [6], part III, p. 58. Theorem 2.2 as stated there is wrong; to make it correct, condition (i') (which essentially coincides with condition (iii) of Theorem 2 in this note) must be omitted. In fact, the class $\boldsymbol{T}$ of torsion groups satisfies conditions (i'), (ii), and (iii) of Theorem 2.2 although, as we noted above, it is not a P_0-class and hence *a fortiori* not an equational class in the sense of [6]. The error was pointed out to the author by Saunders MacLane.

in which denumerably long atomic formulas may occur, and disjunctions and conjunctions of systems of formulas with the power of the continuum may be formed [5].

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[5] We should like to indicate here some other publications in which logical systems with infinitely long expressions are directly or indirectly involved, in fact, [1]. [2], [3], and [4]. In particular, the observations in this note are related to some results in [2]. While the discussion in [2] is lacking a precisely defined logical and set-theoretical basis, it seems that the results of this discussion could be (and probably ought to be) interpreted as belonging to the theory of models for predicate logic $\mathbf{P}_\infty$ with arbitrarily long infinite expressions.

The seminar in the foundations of mathematics conducted by L. Henkin and A. Tarski at the University of California at Berkeley in the fall semester of 1956 was entirely devoted to the discussion of predicate logics with infinitely long expressions. In particular, Henkin and Tarski communicated some new results in this field (not yet published), and Mrs. Carol Karp gave a detailed report on her investigations into the syntax of such logics.

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HOMOLOGICAL RINGOIDS

BY

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1. Introduction. The algebraic study of homology theory may be said to have originated with Poincaré who associated with every compact polyhedron certain numerical invariants, the so-called *Betti numbers* and *torsion coefficients* of the polyhedron [1]. Emmy Noether is credited with the observation that these numerical invariants were in fact invariants of certain finitely-generated Abelian groups, the *homology groups* of the given polyhedron.

More precisely, given any simplicial complex K triangulating a polyhedron $|K|$, one considers, for each dimension n, chains of n-simplexes of the triangulation K, such an n-chain being abstractly an element of the free Abelian group freely generated by the n-simplexes. The boundary of any (oriented) n-simplex is a well-defined $(n-1)$-chain (consisting of the suitably oriented $(n-1)$-faces of the simplex) and one obtains in this way a homomorphism ∂_n from C_n, the group of n-chains, to C_{n-1} [2]. The n-cycles of K are the n-chains in the kernel of ∂_n and the n-boundaries of K are the n-chains in the image of ∂_{n+1}. The fundamental relation $\partial_{n+1}\partial_n = 0$ (homomorphisms are here written on the *right*, so $\partial_{n+1}\partial_n$ means ∂_{n+1} *followed by* ∂_n) implies that the group of n-boundaries $B_n(K)$ is a subgroup of the group of n-cycles $Z_n(K)$ and so a factor group $H_n(K) = Z_n(K)/B_n(K)$ is defined. This factor group is the *n-th homology group* of K and it may be shown that if K, L are triangulations of homeomorphic polyhedra then their homology groups are isomorphic; briefly *the homology groups are topological invariants*. Moreover, the Betti numbers and torsion coefficients of dimension n of the polyhedron $|K|$ are the rank and invariant factors of the finitely-generated Abelian group $H_n(K)$.

To-day the scope of homology theory is very broad. There are various homology theories for general spaces (*e. g.* singular theory, Čech theory); there is a dual theory of cohomology in which additional elements of alge-

[1] It is believed that Heegard pointed out to Poincaré the possibility of torsion in homology relations of cycles.

[2] We may put $C_{-1} = 0$, $\partial_0 = 0$.