autre ensemble, \( M = \{\varphi(x, E)\} \), de fonctions \( \varphi(x, E) \), tel que les deux ensembles \( M \) et \( M_\alpha \) possèdent les propriétés suivantes:

1° Chacune des fonctions \( \varphi(x, E) \), de l’ensemble \( M \) est mesurable et définie presque partout dans l’ensemble correspondant \( E \subset \{-\pi, \pi\} \), mes \( E > 0 \).

2° \( M_\alpha \subset M \).

3° L’ensemble \( M \) est fermé au sens étroit.

4° Quelle que soit la fonction \( \psi(x, e) \), \( \cdot M_\alpha \), il n’existe aucune fonction \( \varphi(x, E) \), \( \cdot M \) pour laquelle \( e \subset E \), mes \( (E - e) > 0 \) et \( \varphi(x, e) = \varphi(x, E) \) presque partout dans \( e \).

5° Quelle que soit la fonction \( \varphi(x, E) \), \( \cdot M_\alpha \), il existe une fonction \( \psi(x, e) \), \( \cdot M_\alpha \) pour laquelle \( E \subset e \) et \( \varphi(x, e) = \varphi(x, E) \) presque partout dans \( E \).

6° Lorsque \( \psi(x, e) \), \( \cdot M_\alpha \) et \( e' \subset e \), mes \( (e - e') = 0 \), la fonction \( \varphi(x, e') \), égale à \( \psi(x, e) \) presque partout dans \( e' \), appartient aussi à l’ensemble \( M_\alpha \).

La question se pose de trouver les conditions nécessaires et suffisantes pour qu’un ensemble \( M = \{\varphi(x, E)\} \) soit celui de toutes les fonctions limites au sens étroit d’une série trigonométrique (4) (P 249).

**Travaux cités**


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**The Sentential Calculus with Infinitely Long Expressions**

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The results given in this note are essentially the observations previously made by the authors concerning the equational identities in Boolean algebras with infinitary operations and in particular the relation between the identities holding in the two-element algebras and those holding in arbitrary Boolean algebras. These remarks, however, are reformulated here in terms of the syntax of the sentential calculus with infinitely long formulas. It is to be noted that the discussion of the sentential calculus is part of a comprehensive study concerning the syntax of the predicate logic with infinitely long expressions which has been undertaken and carried out by Mrs. Carol Karp. The results of Mrs. Karp have not yet been published, but they were presented in the seminar in the foundations of mathematics conducted by L. Henkin and A. Tarski at the University of California at Berkeley in the fall semester of 1956 (1).

Let \( \alpha \) and \( \beta \) be cardinal numbers. (We shall identify the cardinals with the initial ordinals of their respective number classes.) The sentential calculus considered will have \( \beta \) different sentential variables and will permit the formation of well-ordered conjunctions and disjunctions in all lengths less than \( \alpha \). The case where \( \alpha = \beta = \omega \) is simply the ordinary calculus. The case where \( \alpha = \omega \), retains much analogy with the ordinary case and is examined in detail. The cases where \( \alpha > \omega \) and \( \beta > \omega \) present some peculiarities which require the reformulation of the definition of

(1) This note is a summary of a lecture given by the authors at the Summer Institute of Symbolic Logic at Cornell University in July 1957; it appeared under the same title, though in a somewhat shorter version, in *Summaries of talks presented at the Summer Institute of Symbolic Logic in 1957 at Cornell University, vol. 1*, pp. 83-89 (mimeographed). The results of this note were obtained and the note was prepared for publication while Tarski was working on a research project in the foundations of mathematics sponsored by the National Science Foundation. For a Boolean algebraic formulation of the results see the abstracts Scott [7] and Tarski [10]. Some remarks concerning the predicate logic with infinitely long expressions (which is not discussed in this note) can be found in Tarski [11].
a theorem. The exact statement of the result is given, but the proof will be omitted.

As usual there are two aspects of the formal systems to be considered: the syntactic and the semantic. The syntactic details involve the notions of a well-formed formula and a theorem, while the semantic considerations make use of the substitutions of $T$'s and $P$'s for the sentential variables in order to define a tautology. For $\alpha \leq \omega_1$ it is to be shown that a well-formed formula is a theorem if and only if it is a tautology. For $\alpha > \omega_1$, this is generally not the case, unless the notion of theorem is strengthened.

As basic symbols for the calculi we use $\rightarrow, \neg, \wedge, \lor$ standing for implication, negation, conjunction, and disjunction. Brackets [ and ] are also used. The cardinal $\beta$ will be fixed for the discussion, and the symbols $p_0, p_1, \ldots, p_\beta \ldots$ (where $\xi < \beta$) will be used as sentential variables. It is assumed that the notion of transfinite concatenation of symbols is understood. Strictly speaking one should give an explicit set-theoretical construction of the theory of infinite concatenation, but such details present no essential difficulties and are much too lengthy for this note. Concatenation will be denoted here simply by juxtaposition of the symbols.

**Definition 1.** The class of $\alpha$-well-formed formulas (a-wffs) is the least class of formulas closed under the following rules:

(i) a variable $p_\xi$, where $\xi < \beta$, is an a-wff;
(ii) if $A$ and $B$ are a-wffs, then so are $[A \rightarrow B]$ and $\neg A$;
(iii) if $A_0, A_1, \ldots, A_\xi, \ldots$ is a well-ordered sequence of a-wffs of type less than $\alpha$, then $\wedge [A_0, A_1, \ldots, A_\xi, \ldots]$ and $\lor [A_0, A_1, \ldots, A_\xi, \ldots]$ are a-wffs.

Thus, if $\alpha = \omega$, we are considering formulas involving finite conjunctions and disjunctions of arbitrary lengths; while, if $\alpha = \omega_1$, then denumerable conjunctions and disjunctions are permitted. In turn out to be superfluous to consider certain of the cardinals: namely, the so-called singular numbers, i.e., those ordinals that can be written as an ordinal sum of smaller numbers over a smaller index. The first transfinite example among cardinals is of course $\omega_0 = \sum_{\omega} \omega$. Notice that any conjunction of the form $\wedge [A_0, A_1, \ldots, A_\xi, \ldots]$ where $\xi < \omega_0$ should be equivalent in meaning to a conjunction of type $\omega$ of conjunctions of the various types $\omega_\xi$ as follows:

$$\wedge \left[ \wedge [A_0, A_1, \ldots] \wedge [A_0, A_{\xi+1}, \ldots] \wedge \ldots \wedge [A_0, A_{\omega_0}, A_{\omega_0+1}, \ldots] \ldots \right].$$

Hence, there will be an equivalence between the sentential calculi of $\omega_\xi$-wffs and $\omega_{\xi+1}$-wffs. The details of this equivalence for any singular number we leave to the reader and assume henceforth that the cardinal $\alpha$ is regular (i.e., non-singular) and infinite.

In order to simplify the writing of formulas we adopt the following notation:

$$[A \wedge B] \quad \text{for} \quad \wedge [AB],$$

$$[A \lor B] \quad \text{for} \quad \lor [AB],$$

$$\wedge A_\xi \quad \text{for} \quad \wedge [A_0, A_1, A_2, \ldots],$$

$$\lor A_\xi \quad \text{for} \quad \lor [A_0, A_1, A_2, \ldots],$$

where $\gamma$ is the type of the sequence mentioned.

**Definition 2.** The class of $\alpha$-theorems is the least class of a-wffs closed under the following rules:

(i) if $A, B, C$ are a-wffs, then $[A \rightarrow (B \rightarrow A)]$;

(ii) if $A, B, C$ are a-wffs, then $[(A \rightarrow B) \rightarrow (A \rightarrow C)]$ and $[(A \rightarrow B) \rightarrow (A \rightarrow C)]$ are a-theorems;

(iii) if $A_0, A_1, \ldots, A_\xi, \ldots$ is a well-ordered sequence of type $\gamma < \alpha$ of a-wffs, then $[A_0 \rightarrow A_\xi]$ and $[A_\xi \rightarrow A_\eta]$ are a-theorems for each $\eta < \gamma$;

(iv) if $[A \rightarrow B]$ and $A$ are a-theorems, then so is $B$;

(v) if $A_0, A_1, \ldots, A_\xi, \ldots$ is a well-ordered sequence of type $\gamma < \alpha$ of a-wffs $B$ and $C$ is an a-wff, then $[B \rightarrow A_\xi]$ is an a-theorem for all $\eta < \gamma$, then $[B \rightarrow A_\xi]$ is an a-theorem;

(vi) if $[B \rightarrow A_\xi]$ is an a-theorem for all $\eta < \gamma$, then $[B \rightarrow A_\xi]$ is an a-theorem.

**Definition 3.** Let $\gamma < \alpha$. A class $K$ of $\alpha$-wffs is $\gamma$-consistent if and only if there is no sequence $A_0, A_1, \ldots, A_\xi, \ldots$ of elements of $K$ of type $\gamma' < \gamma$ such that $\sim \wedge A_\xi$ is an a-theorem.

In the ordinary calculus with $\alpha = \omega$ the ordinary notion of consistency is what we would here call $\omega_0$-consistency. Actually, the only notion used below in the proofs is that of $\omega_0$-consistency, for which we now state a fundamental property.

**Lemma 1.** If $A$ is an a-wff that is not an a-theorem, and if $B_0, i < \omega$ and $\xi < \omega_1 < \alpha$ is a double sequence of a-wffs, then there exist functions

(1) Rules (i) and (iii) supply us with the theorems of the ordinary sentential calculus, and hence the class of all a-wffs can be divided into equivalence classes forming a Boolean algebra. Rules (ii) and (iv) assure us that the algebra is complete in all degrees less than $\alpha$. If $\alpha = \omega_1$, for example, we obtain the free Boolean algebra with $\beta$ generators.
\( \varphi \text{ and } \psi \text{ such that } \varphi(i) < \alpha_i \text{ and } \psi(i) < \alpha_i \text{ for all } i < \omega \) and the class of formulas

\[
\neg \lambda \top \lor (\bigwedge_{t \in \mathbb{N}} B_t[k] < \omega) \lor (\bigvee_{t \in \mathbb{N}} B_t[k] > B_{t+1})[i < \omega]
\]

is \( \alpha \), \( \alpha \)-consistent \((\dagger)\).

Before we can define the notion of a tautology, it must be verified that \( T \)'s and \( F \)'s can be substituted for the sentential variables in the proper way. Proceeding by a transfinite induction based on Definition 3, the following lemma should be proved:

**Lemma 2.** If the function \( f \) is a substitution defined on the variables \( p_2 \), taking the values \( T \) and \( F \), then there is a unique extension \( f' \) of \( f \) to the class of all \( \alpha \)-wffs such that

(i) \( f' \) takes on only the values \( T \) and \( F \);

(ii) \( f'(A \rightarrow B) = T \) if and only if either \( f'(A) = F \) or \( f'(B) = T \);

(iii) \( f'(-A) = T \) if and only if \( f'(A) = F \);

(iv) \( f'\left(\bigwedge_{t \in \mathbb{N}} A_t\right) = T \) if and only if \( f'(A_t) = T \) for all \( t \leq \gamma \);

(v) \( f'\left(\bigvee_{t \in \mathbb{N}} A_t\right) = T \) if and only if \( f'(A_t) = T \) for some \( t \leq \gamma \).

**Definition 4.** An \( \alpha \)-wff \( \lambda \) is an \( \alpha \)-tautology if and only if \( f'(\lambda) = T \) for all substitutions \( f \).

**Theorem 1** (Completeness Theorem for Countable Calculus). If \( \alpha \leq \alpha_1 \), then an \( \alpha \)-wff is an \( \alpha \)-wff if and only if it is an \( \alpha \)-tautology \((\dagger)\).

Proof (in outline). It is obvious from the definitions that every \( \alpha \)-wff is an \( \alpha \)-wff. Assume then that \( \lambda \) is an \( \alpha \)-wff that is not an \( \alpha \)-wff. Let \( S \) be the least class of \( \alpha \)-wffs containing \( \lambda \) and closed under the operation of taking subformulas. The essential point of the argument is that \( S \) is at most denumerable as a consequence of the hypothesis \( \alpha \leq \alpha_1 \). Thus there exists a sequence \( B_t[k] < \omega \) and \( \xi < \alpha_t < \alpha_1 \) such that all conjunctions and disjunctions in the class \( S \) are contained in the class

\[
\left(\bigwedge_{t \in \mathbb{N}} B_t[k] < \omega\right) \lor \left(\bigvee_{t \in \mathbb{N}} B_t[k] > B_{t+1}\right)[i < \omega].
\]

Apply next Lemma 1 to obtain the \( \alpha \), \( \alpha \)-consistent class mentioned in the conclusion of Lemma 1. With the aid of some form of the axiom

\[(\dagger)\] This is a syntactic formulation of the lemma of Łosowa-Sikorski \([5]\) (p. 197, statement (iv)). The proof given there is topological, but a simple inductive proof (credited to Tarski) is given in the review by Feferman \([3]\).

\[(\dagger)\] In Boolean algebraic terms, this theorems in effect shows that the free Boolean \( \alpha \)-algebra is isomorphic to a \( \alpha \)-field of sets, a statement equivalent to the theorem of Łomos \([4]\). See also Rieger \([6]\) and Sikorski \([9]\). Thus, Theorem 1 is not essentially new.

of choice \((\dagger)\) obtain an extension of that \( \alpha \), \( \alpha \)-consistent class to a maximal \( \alpha \), \( \alpha \)-consistent class \( M \). Define a substitution \( f \) on the variables \( p_t \) by the condition that \( f(p_t) = T \) if and only if \( p_t \) is in the class \( M \). It is then finally to be verified that \( t'(\lambda) = T \), and hence \( A \) is not a tautology, as was to be shown.

Let us turn now to the case where \( \alpha > \alpha_1 \). Of course if \( \beta \) were finite, there would be only a finite number of inequivalent formulas on the basis of the axioms given in Definition 2. Whence, all infinite operations could be eliminated in favor of the finite ones. Thus, assume that \( \beta > \omega \). That the axioms given above are inadequate for generating all tautologies is shown by the following counter-example: Let \( \alpha \) actually be of greater power than that of the continuum. Let \( p_t \), where \( \xi < \gamma \), be a well-ordering of all functions from the integers to \([0,1] \). Then the formula

\[
\left(\bigwedge_{t \in \mathbb{N}} p_t \lor p_{t+1} \right) \rightarrow \left(\bigvee_{t \in \mathbb{N}} p_t \lor p_{t+1+\xi}\right)
\]

is an \( \alpha \)-tautology that is not an \( \alpha \)-wff \((\dagger)\). It would seem still to be an open question whether such a counter-example can be given for all \( \beta > \alpha_1 \) without the aid of the continuum hypothesis. Nevertheless, an adequate axioms system can be given for all cardinals \( \alpha \) by adding an additional clause to the definition of \( \alpha \)-wffs.

**Definition 5.** The class of \( \alpha \)-theorems in the strict sense is the least class of \( \alpha \)-wffs closed under the rules of Definition 2 as well as the following additional rules:

\[(\ddagger)\] This theorems is the definition of the completeness theorem can be proved in the following form \((\dagger)\):

\[(\ddagger)\] This counter-example is taken from Sikorski \([8]\).
THEOREM 2 (COMPLETE THEOREM FOR ARBITRARY CALCULI). If \( a \) is any cardinal number, then an \( a \)-thick is an \( a \)-theorem in the strict sense if and only if it is an \( a \)-tautology.

It is well-known that in the case \( a = \omega \) there is an even stronger completeness result: any \( \omega \), \( \omega \)-consistent class of \( \omega \)-wffs has a substitution giving all formulas in the class the value \( T \). For all \( a > \omega \) it can be shown that such a stronger result fails unless possibly \( a \) is a strongly inaccessible cardinal number. Whether the stronger theorem is true in the inaccessible case is an open question (P 250) seemingly involving fundamental set-theoretical problems (6). However, certain stronger results are possible: for example, it should be clear from the proof of Theorem 1 that every at most denumerable and \( \omega, \omega \)-consistent class of \( \omega \)-wffs has a substitution giving all formulas in the class the value \( T \).

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As extensions of ordinary first order predicate logic \( P \), various systems of predicate logic with infinitely long expressions can be considered (1).

To fix the ideas we restrict ourselves to the discussion of predicate logic \( P \), with denumerably long expressions. Atomic formulas in \( P \) are expressions like

\[ \varphi(v_1, \ldots, v_n) \]

consisting of a predicate \( \varphi \) and a finite sequence of variables \( \langle v_1, \ldots, v_n \rangle \).

The set of all predicates is assumed to be at most denumerable (though this restriction is not essential) and to contain the binary identity predicate \( = \); instead of \( = (v_0, v_1) \) we write \( v_0 = v_1 \). Compound well-formed formulas are obtained from simpler ones by means of the following operations: (i) the formation of the negation \( \neg F \) of a formula \( F \); (ii) the formation of the implication \( [F \rightarrow F'] \) of two formulas \( F \) and \( F' \); (iii) the formation of the disjunction

\[ \bigvee [F_0, F_1, \ldots] \]

and the conjunction

\[ \bigwedge [F_0, F_1, \ldots] \]

of a finite or denumerable sequence of formulas \( \langle F_0, F_1, \ldots \rangle \); (iv) the universal quantification

\[ (\forall v_0, \ldots, v_n) F \]

(1) This note contains the text of the remarks made by the author at the Summer Institute of Symbolic Logic in 1957 at Cornell University; it first appeared (under the same title, though in a more concise form) in Summaries of talks presented at the Summer Institute of Symbolic Logic in 1957 at Cornell University, vol. 1, p. 160-163 (mimeographed). The results of this note were obtained and the note was prepared for publication while the author was working on a research project in the foundations of mathematics sponsored by the National Science Foundation and carried through in the University of California, Berkeley.