autre ensemble, $M = \{\varphi(x, E)\}$, de fonctions $\varphi(x, E)$, tel que les deux ensembles M et M_0 possèdent les propriétés suivantes:

1° *Chacune des fonctions $\varphi(x, E)$ de l'ensemble M est mesurable et définie presque partout dans l'ensemble correspondant $E \subset [-\pi, \pi]$,* $\operatorname{mes} E > 0$.

2° $M_0 \subset M$.

3° *L'ensemble M est fermé au sens étroit.*

4° *Quelle que soit la fonction $\psi(x, e) \in M_0$, il n'existe aucune fonction $\varphi(x, E) \in M$ pour laquelle $e \subset E$, $\operatorname{mes}(E-e) > 0$ et $\psi(x, e) = \varphi(x, E)$ presque partout dans e.*

5° *Quelle que soit la fonction $\varphi(x, E) \in M$, il existe une fonction $\psi(x, e) \in M_0$ pour laquelle $E \subset e$ et $\psi(x, e) = \varphi(x, E)$ presque partout dans E.*

6° *Lorsque $\psi(x, e) \in M_0$ et $e' \subset e$, $\operatorname{mes}(e-e') = 0$, la fonction $g(x, e')$, égale à $\psi(x, e)$ presque partout dans e', appartient aussi à l'ensemble M_0.*

La question se pose de trouver les conditions nécessaires et suffisantes pour qu'un ensemble $M = \{\varphi(x, E)\}$ soit celui de toutes les fonctions limites au sens étroit d'une série trigonométrique (4) (**P 249**).

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THE SENTENTIAL CALCULUS WITH INFINITELY LONG EXPRESSIONS

BY

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The results given in this note are essentially the observations previously made by the authors concerning the equational identities in Boolean algebras with infinitary operations and in particular the relation between the identities holding in the two-element algebras and those holding in arbitrary Boolean algebras. These remarks, however, are reformulated here in terms of the syntax of the sentential calculus with infinitely long formulas. It is to be noted that the discussion of the sentential calculus is part of a comprehensive study concerning the syntax of the predicate logic with infinitely long expressions which has been undertaken and carried out by Mrs. Carol Karp. The results of Mrs. Karp have not yet been published, but they were presented in the seminar in the foundations of mathematics conducted by L. Henkin and A. Tarski at the University of California at Berkeley in the fall semester of 1956 (1).

Let α and β be cardinal numbers. (We shall identify the cardinals with the initial ordinals of their respective number classes). The sentential calculi considered will have β different sentential variables and will permit the formation of well-ordered conjunctions and disjunctions in all lengths less than α. The case where $\alpha = \beta = \omega$ is simply the ordinary calculus. The case where $\alpha = \omega_1$ retains much analogy with the ordinary case and is examined in detail. The cases where $\alpha > \omega_1$ and $\beta \geqslant \omega$ present some peculiarites which require the reformulation of the definition of

(1) This note is a summary of a lecture given by the authors at the Summer Institute of Symbolic Logic at Cornell University in July 1957; it appeared under the same title, though in a somewhat shorter version, in *Summaries of talks presented at the Summer Institute of Symbolic Logic in 1957 at Cornell University*, vol. 1, p. 83-89 (mimeographed). The results of this note were obtained and the note was prepared for publication while Tarski was working on a research project in the foundations of mathematics sponsored by the National Science Foundation. For a Boolean algebraic formulation of the results see the abstracts Scott [7] and Tarski [10]. Some remarks concerning the predicate logic with infinitely long expressions (which is not discussed in this note) can be found in Tarski [11].

a theorem. The exact statement of the result is given, but the proof will be omitted.

As usual there are two aspects of the formal systems to be considered: the syntactic and the semantic. The syntactic details involve the notions of a well-formed formula and a theorem, while the semantic considerations make use of the substitutions of T's and F's for the sentential variables in order to define a tautology. For $a \leqslant \omega_1$ it is to be shown that a well-formed formula is a theorem if and only if it is a tautology. For $a > \omega_1$ this is generally not the case, unless the notion of theorem is strengthened.

As basic symbols for the calculi we use $\rightarrow, \sim, \wedge, \vee$ standing for implication, negation, conjunction, and disjunction. Brackets [and] are also used. The cardinal β will be fixed for the discussion, and the symbols $p_0, p_1, \ldots p_\xi, \ldots$ (where $\xi < \beta$) will be used as sentential variables. It is assumed that the notion of transfinite concatenation of symbols is understood. Strictly speaking one should give an explicit set-theoretical construction of the theory of infinite concatenation, but such details present no essential difficulties and are much too lengthy for this note. Concatenation will be denoted here simply by juxtaposition of the symbols.

Definition 1. *The class of* ***a-well-formed formulas*** *(a-**wffs**) is the least class of formulas closed under the following rules:*

(i) *a variable p_ξ, where $\xi < \beta$, is an a-wff;*

(ii) *if A and B are a-wffs, then so are $[A \rightarrow B]$ and $\sim A$;*

(iii) *if $A_0, A_1, \ldots, A_\xi, \ldots$ is a well-ordered sequence of a-wffs of type less than a, then $\wedge [A_0 A_1 \ldots A_\xi \ldots]$ and $\vee [A_0 A_1 \ldots A_\xi \ldots]$ are a-wffs.*

Thus, if $a = \omega$, we are considering formulas involving finite conjunctions and disjunctions of arbitrary lengths; while, if $a = \omega_1$, then denumerable conjunctions and disjunctions are permitted. In turns out to be superfluous to consider certain of the cardinals: namely, the so-called singular numbers, *i. e.*, those ordinals that can be written as an ordinal sum of smaller numbers over a smaller index. The first transfinite example among cardinals is of course $\omega_\omega = \sum_{n<\omega} \omega_n$. Notice that any conjunction of the form $\wedge [A_0 A_1 \ldots A_\xi \ldots]$ where $\xi < \omega_\omega$ should be equivalent in meaning to a conjunction of type ω of conjunctions of the various types ω_n as follows:

$$\wedge [\wedge [A_0 A_1 \ldots] \wedge [A_\omega A_{\omega+1} \ldots] \wedge \ldots \wedge [A_{\omega_n} A_{\omega_n+1} \ldots] \ \ldots].$$

Hence, there will be an equivalence between the sentential calculi of ω_ω-wffs and $\omega_{\omega+1}$-wffs. The details of this equivalence for any singular

number we leave to the reader and assume henceforth that the cardinal a is regular (*i. e.* non-singular) and infinite.

In order to simplify the writing of formulas we adopt the following notation:

$$\begin{array}{lcl} [A \wedge B] & \text{for} & \wedge [AB], \\ [A \vee B] & \text{for} & \vee [AB], \\ \bigwedge_{\xi<\gamma} A_\xi & \text{for} & \wedge [A_0 A_1 \ldots A_\xi \ldots], \\ \bigvee_{\xi<\gamma} A_\xi & \text{for} & \vee [A_0 A_1 \ldots A_\xi \ldots], \end{array}$$

where γ is the type of the sequence mentioned.

Definition 2. *The class of* ***a-theorems*** *is the least class of a-wffs closed under the following rules:*

(i) *if A, B, C are a-wffs, then $[A \rightarrow [B \rightarrow A]]$, $\big[[A \rightarrow [B \rightarrow C]] \rightarrow [[A \rightarrow B] \rightarrow [A \rightarrow C]]\big]$ and $[[\sim B \rightarrow \sim A] \rightarrow [A \rightarrow B]]$ are a-theorems;*

(ii) *if $A_0, A_1, \ldots, A_\xi, \ldots$ is a well-ordered sequence of type $\gamma < a$ of a-wffs, then $[\bigwedge_{\xi<\gamma} A_\xi \rightarrow A_\eta]$ and $[A_\eta \rightarrow \bigvee_{\xi<\gamma} A_\xi]$ are a-theorems for each $\eta < \gamma$;*

(iii) *if $[A \rightarrow B]$ and A are a-theorems, then so is B;*

(iv) *if $A_0, A_1 \ldots, A_\xi, \ldots$ is a well-ordered sequence of type $\gamma < a$ of a-wffs and B is an a-wff, then*

(iv') *if $[B \rightarrow A_\eta]$ is an a-theorem for all $\eta < \gamma$, then $[B \rightarrow \bigwedge_{\xi<\gamma} A_\xi]$ is an a-theorem;*

(iv'') *if $[A_\eta \rightarrow B]$ is an a-theorem for all $\eta < \gamma$, then $[\bigvee_{\xi<\gamma} A_\xi \rightarrow B]$ is an a-theorem* (2).

Definition 3. *Let $\gamma \leqslant a$. A class K of a-wffs is γ, a-**consistent** if and only if there is no sequence $A_0, A_1, \ldots, A_\xi, \ldots$ of elements of K of type $\gamma' < \gamma$ such that $\sim \bigwedge_{\xi<\gamma'} A_\xi$ is an a-theorem.*

In the ordinary calculus with $a = \omega$ the ordinary notion of consistency is what we would here call ω, *ω-consistency*. Actually, the only notion used below in the proofs is that of ω, a-consistency, for which we now state a fundamental property.

Lemma 1. *If A is an a-wff that is not an a-theorem, and if $B_{i\xi}$, $i < \omega$ and $\xi < a_i < a$, is a double sequence of a-wffs, then there exist functions*

(2) Rules (i) and (iii) supply us with the theorems of the ordinary sentential calculus, and hence the class of all a-wffs can be divided into equivalence classes forming a Boolean algebra. Rules (ii) and (iv) assure us that the algebra is complete in all degrees less than a. If $a = \omega_1$, for example, we obtain the free Boolean σ-algebra with β generators.

φ and ψ such that $\varphi(i) < a_i$ and $\psi(i) < a_i$ for all $i < \omega$ and the class of formulas

$$\{\sim A\} \cup \{[B_{i\varphi(i)} \to \bigwedge_{\xi<a_i} B_{i_\xi}] | i < \omega\} \cup \{[\bigvee_{\xi<a_i} B_{i_\xi} \to B_{i\psi(i)}] | i < \omega\}$$

is ω, α-consistent (3).

Before we can define the notion of a tautology, it must be verified that T's and F's can be substituted for the sentential variables in the proper way. Proceeding by a transfinite induction based on Definition 1, the following lemma should be proved:

LEMMA 2. *If the function f is a substitution defined on the variables p_ξ taking the values T and F, then there is a unique extension f^* of f to the class of all α-wffs such that*

(i) *f^* takes on only the values T and F;*
(ii) *$f^*([A \to B]) = T$ if and only if either $f^*(A) = F$ or $f^*(B) = T$;*
(iii) *$f^*(\sim A) = T$ if and only if $f^*(A) = F$;*
(iv) *$f^*(\bigwedge_{\xi<\gamma} A_\xi) = T$ if and only if $f^*(A_\xi) = T$ for all $\xi < \gamma$;*
(v) *$f^*(\bigvee_{\xi<\gamma} A_\xi) = T$ if and only if $f^*(A_\xi) = T$ for some $\xi < \gamma$.*

DEFINITION 4. *An α-wff A is an α-**tautology** if and only if $f^*(A) = T$ for all substitutions f.*

THEOREM 1 (COMPLETENESS THEOREM FOR COUNTABLE CALCULI). *If $\alpha \leqslant \omega_1$, then an α-wff is an α-theorem if and only if it is an α-tautology* (4).

Proof (in outline). It is obvious from the definitions that every α-theorem is an α-tautology. Assume then that A is an α-wff that is not an α-theorem. Let S be the least class of α-wffs containing A and closed under the operation of taking subformulas. The essential point of the argument is that S is at most denumerable as a consequence of the hypothesis $\alpha \leqslant \omega_1$. Thus there exists a sequence B_{i_ξ}, $i < \omega$ and $\xi < a_i < \alpha$, such that all conjunctions and disjunctions in the class S are contained in the class

$$\{\bigwedge_{\xi<a_i} B_{i_\xi} | i < \omega\} \cup \{\bigvee_{\xi<a_i} B_{i_\xi} | i < \omega\}.$$

Apply next Lemma 1 to obtain the ω, α-consistent class mentioned in the conclusion of Lemma 1. With the aid of some form of the axiom of choice (5) obtain an extension of that ω, α-consistent class to a maximal ω, α-consistent class M. Define a substitution f on the variables p_ξ by the condition that $f(p_\xi) = T$ if and only if p_ξ is in the class M. It is then finally to be verified that $f^*(A) = F$, and hence A is not a tautology, as was to be shown.

(3) This is a syntactic formulation of the lemma of Rasiowa-Sikorski [5] (p. 197, statement (iv)). The proof given there is topological, but a simple inductive proof (credited to Tarski) is given in the review by Feferman [3].

(4) In Boolean algebraic terms, this theorem in effect shows that the free Boolean σ-algebra is isomorphic to a σ-field of sets, a statement equivalent to the theorem of Loomis [4]. See also Rieger [6] and Sikorski [9]. Thus, Theorem 1 is not essentially new.

Let us turn now to the case where $\alpha > \omega_1$. Of course if β were finite, there would be only a finite number of inequivalent formulas on the basis of the axioms given in Definition 2. Whence, all infinite operations could be eliminated in favor of the finite ones. Thus, assume that $\beta \geqslant \omega$. That the axioms given above are inadequate for generating all tautologies is shown by the following counter-example: Let α actually be of greater power than that of the continuum. Let φ_ξ, where $\xi < \gamma$, be a well-ordering of all functions from the integers to $\{0,1\}$. Then the formula

$$[\bigwedge_{n<\omega} [p_{2n} \vee p_{2n+1}] \to \bigvee_{\xi<\gamma} \bigwedge_{n<\omega} p_{2n+\varphi_\xi(n)}]$$

is an α-tautology that is not an α-theorem (6). It would seem still to be an open question whether such a counter-example can be given for all $\alpha > \omega_1$ without the aid of the continuum hypothesis. Nevertheless, an adequate axiom system can be given for all cardinals α by adding an additional clause to the definition of α-theorems.

DEFINITION 5. *The class of α-**theorems in the strict sense** is the least class of α-wffs closed under the rules of Definition 2 as well as the following additional rule:*

(v) *if $\gamma < \alpha$ and $A_{\xi\eta}$, with $\xi, \eta < \gamma$, is a double sequence of α-wffs such that for any function φ on γ into γ there exist $\xi, \xi' < \gamma$ such that $A_{\xi\varphi(\xi)} = \sim A_{\xi'\varphi(\xi')}$, then $\bigvee_{\xi<\gamma} \bigwedge_{\xi<\eta} A_{\xi\eta}$ is an α-theorem.*

Under the new definition the completeness theorem can be proved in the following form (7):

(5) The full axiom of choice is not really used here. The only consequence needed is that the ordinal ω_1 is regular, a fact essential to the proof that the set S is at most denumerable. Then since the sequence of variables is given as a well-ordered sequence, the required substitution f can be defined by transfinite recursion by successively obtaining consistent adjunctions of p_ξ or $\sim p_\xi$ to the set constructed in Lemma 1.

(6) This counter-example is taken from Sikorski [8].

(7) Theorem 2 does seem to be new. In Boolean algebraic terms it yields an explicit equational definition of those Boolean algebras which are representable as a field of sets complete in all degrees less than α modulo an ideal complete in all degrees less than α. The proof is based directly on the work of Chang [1]; see especially p. 209, Definition, and p. 211, Theorem 2.

THEOREM 2 (COMPLETENESS THEOREM FOR ARBITRARY CALCULI). *If a is any cardinal number, then an a-wff is an a-theorem in the strict sense if and only if it is an a-tautology.*

It is well-known that in the case $a = \omega$ there is an even stronger completeness result: any ω, ω-consistent class of ω-wffs has a substitution giving all formulas in the class the value T. For all $a > \omega$ it can be shown that such a stronger result fails unless possibly a is a strongly inaccessible cardinal number. Whether the stronger theorem is true in the inaccessible case is an open question (P 250) seemingly involving fundamental set-theoretical problems (8). However, certain stronger results are possible: for example, it should be clear from the proof of Theorem 1 that every at most denumerable and ω, ω_1-consistent class of ω_1-wffs has a substitution giving all formulas in the class the value T.

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(8) This problem is directly related to those problems about inaccessible numbers formulated at the end of Erdös-Tarski [2].

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REMARKS ON PREDICATE LOGIC WITH INFINITELY LONG EXPRESSIONS

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As extensions of ordinary first order predicate logic P_0 various systems of predicate logic with infinitely long expressions can be considered (1).

To fix the ideas we restrict ourselves to the discussion of predicate logic P_1 with denumerably long expressions. Atomic formulas in P_1 are expressions like

$$\varphi(v_0 \ldots v_{n-1})$$

consisting of a predicate φ and a finite sequence of variables $\langle v_0, \ldots, v_{n-1} \rangle$. The set of all predicates is assumed to be at most denumerable (though this restriction is not essential) and to contain the binary identity predicate $=$; instead of $=(v_0, v_1)$ we write $(v_0 = v_1)$. Compound well-formed formulas are obtained from simpler ones by means of the following operations: (i) the formation of the negation $\sim F$ of a formula F; (ii) the formation of the implication $[F_0 \to F_1]$ of two formulas F_0 and F_1; (iii) the formation of the disjunction

$$\bigvee [F_0 \ldots F_\xi \ldots]$$

and the conjunction

$$\bigwedge [F_0 \ldots F_\xi \ldots]$$

of a finite or denumerable sequence of formulas $\langle F_0, \ldots, F_\xi, \ldots \rangle$; (iv) the universal quantification

$$(\forall v_0 \ldots v_\xi \ldots) F$$

(1) This note contains the text of the remarks made by the author at the Summer Institute of Symbolic Logic in 1957 at Cornell University; it first appeared (under the same title, though in a more concise form) in *Summaries of talks presented at the Summer Institute of Symbolic Logic in 1957 at Cornell University*, vol. 1, p. 160-163 (mimeographed). The results of this note were obtained and the note was prepared for publication while the author was working on a research project in the foundations of mathematics sponsored by the National Science Foundation and carried through in the University of California, Berkeley.