COLLOQUIUM MATHEMATICUM

VOL. VI DÉDIÉ À M. CASIMIR KURATOWSKI 1958

ON MAPPINGS OF PLANE SETS

BY

J. L. KELLEY (CAMBRIDGE)

The question of the possible existence of a fixed point free continuous mapping of a plane, non-separating, continuum into itself is an old unsolved problem. The same problem for a periodic transformation of a connected, not necessarily closed, set which fails to cut the plane is also unsolved. The purpose of this note is to phrase two conjectures which imply positive solutions to these problems, and to establish simple special cases of the conjectures. It is hoped that the approach may be suggestive.

CONJECTURE A. *If f is a continuous fixed point free periodic map of a connected plane set into itself then the map $z \to f(z)-z$ is essential in the sense of Kuratowski* [2] *and Eilenberg* [1] **(P 247)**.

For a map of period two we have: If $z \to f(z)-z$ is inessential then $f(z)-z = \exp q(z)$ for some continuous complex valued function q. Then $ff(z)-f(z) = z-f(z) = \exp qf(z)$, and therefore $-1 = [z-f(z)]/[f(z)-z] = \exp[qf(z)-q(z)]$. Hence $qf(z)-q(z) = (2n+1)\pi$ for some integer n and for all z in the connected set. But, replacing z by $f(z)$ in the preceding, we have $q(z)-qf(z) = (2n+1)\pi$, whence $(2n+1)\pi = -(2n+1)\pi$, which is a contradiction. Hence (A) holds for maps of period two.

CONJECTURE B. *If f is a continuous fixed point free mapping of the plane continuum X into itself, then the map $z \to f(z)-z$ is essential* **(P 248)**.

In case X is a neighborhood retract of the union of X and the unbounded component U of the complement of X, we may establish B as follows:

Suppose that the map $z \to f(z)-z$ is inessential, and that r is a continuous retraction onto X of a neighborhood of X in $X \cup U$. Since $\inf\{|f(z)-z| : z \in X\} > 0$, the composition fr is fixed point free on a suitably small neighborhood V of X in $X \cup U$, and, since $z \to fr(z)-r(z)$ is inessential on V, we may suppose that $z \to fr(z)-z$ is inessential on V. Without great difficulty it may be seen that V can be taken closed and such that the boundary B of V in $X \cup U$ is a rectilinear Jordan curve. There is then a continuous deformation $(z, t) \to q_t(z)$ of the plane such that for each t, $0 \leqslant t \leqslant 1$, q_t is a homeomorphism of the plane onto itself, q_0 is the identity mapping, and q_1 carries B onto the circumference $C = \{z: |z| = 1\}$. Because the mapping $z \to fr(z) - z = q_0 fr(z) - q_0(z)$ is an inessential map of X, so is $z \to q_1 fr(z) - q_1(z)$, and hence the map $w \to q_1 fr q_1^{-1}(w) - w$ is an inessential map of $q_1[X]$. Letting s denote the mapping $q_1 fr q_1^{-1}$ restricted to the circumference C, we see that s is continuous and fixed point free, carrying C into $\{z: |z| \leqslant 1\}$, and that $w \to s(w) - w$ is inessential on C. But since $s(w)$ belongs to the disk $\{z: |z| \leqslant 1\}$, the argument of $[s(w) - w]/[-w]$ is in absolute value less than $\pi/2$. It follows that $w \to [s(w) - w]/[-w]$ is inessential, and hence $w \to -w$ is inessential. But this is a contradiction, since it is known that the latter mapping is essential.

REFERENCES

[1] S. Eilenberg, *Transformations continues en circonférence et la topologie du plan*, Fundamenta Mathematicae 26 (1936), p. 60-112.

[2] C. Kuratowski, *Théorèmes sur l'homotopie des fonctions continues de variable complexe et leurs rapports à la théorie des fonctions analytiques*, ibidem 33 (1945), p. 316-367.

UNIVERSITY OF CALIFORNIA, CAMBRIDGE UNIVERSITY

Reçu par la Rédaction le 4. 12. 1957

COLLOQUIUM MATHEMATICUM

VOL. VI DÉDIÉ À M. CASIMIR KURATOWSKI 1958

SUR LES SUITES CONVERGENTES DES SOMMES PARTIELLES DES SÉRIES TRIGONOMÉTRIQUES

PAR

D. MENCHOFF (MOSCOU)

Soit

$$\sum_{n=0}^{\infty} u_n(x) \tag{1}$$

une série dont les termes sont des fonctions mesurables, finies presque partout dans un segment $[a, b]$. Posons

$$Q_n(x) = \sum_{\nu=0}^{n} u_\nu(x) \qquad (n = 0, 1, 2, \ldots) \tag{2}$$

et introduisons la définition suivante:

Définition 1. Nous dirons qu'une fonction $\varphi(x) \equiv \varphi(x, E)$, définie presque partout dans un ensemble $E \subset [a, b]$ [1] de mesure positive, est une *fonction limite de la série* (1) *sur l'ensemble* E s'il existe une suite croissante de nombres entiers et positifs ϱ_k, $k = 1, 2, \ldots$, telle que

$$\lim_{k\to\infty} Q_{\varrho_k}(x) = \varphi(x) \tag{3}$$

presque partout dans E.

Considérons un ensemble $M = \{\varphi(x)\}$ de fonctions mesurables $\varphi(x)$ dont chacune est définie dans un ensemble $E \subset [a, b]$ de mesure positive. On peut obtenir la condition nécessaire et suffisante pour que l'ensemble M soit celui de toutes les fonctions limites d'une série trigonométrique

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{4}$$

sur l'ensemble E.

[1] En disant qu'une fonction $\varphi(x)$ est définie presque partout dans un ensemble E, nous admettons la possibilité que $\varphi(x) = +\infty$ ou $\varphi(x) = -\infty$ sur un ensemble de mesure positive.