For the error estimate we have the formula (16).

Remark 3. Remarks 1 and 2 are also applicable to Theorem 2.

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ON CANTOR'S PRODUCTS

BY

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G. Cantor [4] (see also [16], p. 122-127) considered the representation of a real number $x>1$ in the form of the infinite product

$$x = \prod_{n=1}^{\infty}\left(1+\frac{1}{q_n}\right) \tag{1}$$

where $q_n = q_n(x)$ is a sequence of positive integers, which may be defined as follows: we choose for q_1 the least positive integer for which $1+1/q_1 \leqslant x$ and if $q_1, q_2, \ldots, q_{n-1}$ are already chosen, we choose for q_n the least positive integer for which $\prod_{k=1}^{n}(1+1/q_k) \leqslant x$. Clearly if x is contained in the interval $2^{k-1} < x \leqslant 2^k$ $(k = 1, 2 \ldots)$, then $q_1 = q_2 = \ldots = q_{k-1} = 1$, and $1 < x/\prod_{j=1}^{k-1}(1+1/q_j) \leqslant 2$. Thus we may restrict ourselves to the values of x lying in the interval $1 < x \leqslant 2$. In this case clearly

$$q_{n+1} \geqslant q_n^2 \quad (n = 1, 2, \ldots). \tag{2}$$

Let us put

$$E_0(x) = x, \quad E_n(x) = x\Big/\prod_{k=1}^{n}\left(1+\frac{1}{q_k}\right) \quad (n = 1, 2, \ldots). \tag{3}$$

It is easy to see that if x is rational, $x = a/b$ where a and b are positive integers, $b < a \leqslant 2b$, then we obtain by the algorithm described above a finite representation for x of the form

$$\frac{a}{b} = \prod_{n=1}^{N}\left(1+\frac{1}{q_n}\right) \tag{4}$$

since putting $E_n(a/b) = a_n/b_n$ we have $a_{n+1}-b_{n+1} < a_n - b_n$; it follows that $N \leqslant a-b$.

If x in irrational, then, by (2), q_n tends to $+\infty$ for $n\to\infty$, and since

$$1 \leqslant E_n(x) \leqslant 1+\frac{1}{q_n^2-1}, \tag{5}$$

it follows that $\lim_{n\to\infty} E_n(x)=1$. This implies the validity of (1). For irrational values of x clearly strict inequality in (2) stands for an infinity of values of n, because by the identity

$$\prod_{\nu=0}^{\infty}(1+x^{2^\nu})=\frac{1}{1-x} \qquad (|x|<1); \tag{6}$$

if equality in (2) stood for $n\geqslant n_0$, then x would be rational.

In the present paper we consider the asymptotic behaviour of the sequence $q_n=q_n(x)$ by using the methods of probability theory.

The other classical representations of real numbers have already been investigated from this point of view. For q-adic expansions (including decimal fractions) the results of É. Borel [1] and D. Raikov [17], for continued fractions the results of R. O. Kuzmin [13], A. O. Khintchine [9], [10], [11], P. Lévy ([15], Chapitre IX, p. 290), and C. Ryll-Nardzewski [10] (see also [7] and [8]) are well known. Recently [18] I have extended these results to a general class of representations (including q-adic expansions and continued fractions as special cases) called "f-expansions" and having the form

$$x=f(\varepsilon_1+f(\varepsilon_2+f(\varepsilon_3+\ldots)\ldots) \tag{7}$$

where $\varepsilon_1, \varepsilon_2, \ldots$ are non-negative integers.

Engel's series have been investigated from a probabilistic point of view by É. Borel [2], [3], P. Lévy [14] and recently by P. Erdős, P. Szüsz and the author of the present paper [5]. In [5] the statistical properties of Sylvester's series

$$x=\frac{1}{Q_1}+\frac{1}{Q_2}+\ldots+\frac{1}{Q_n}+\ldots \qquad (0<x<1) \tag{8}$$

(where $Q_1, Q_2, \ldots$ are natural numbers $\geqslant 2$ and $Q_{n+1}\geqslant Q_n(Q_n-1)+1$ for $n=1,2,\ldots$) are also considered.

It seems, however, that Cantor's products have not been considered up to now from the point of view of probability theory.

The aim of the present paper is to fill this gap. Thus we shall consider the functions $q_n=q_n(x)$ $(1<x\leqslant 2)$ as random variables on the probability space, furnished by the interval $(1,2]$ and the Lebesgue-measure on it. In other words, we interpret the Lebesgue-measure of the set of

those real numbers x for which some relation concerning the values of $q_n(x)$ holds as the probability of this relation and denote it by $P(\ldots)$ where in the brackets the relation in question is indicated.

First we prove the following

LEMMA 1. *The random variables*

$$\xi_n=(q_n^2(x)-1)(E_n(x)-1) \qquad (n=1,2,\ldots) \tag{9}$$

are all uniformly distributed in the interval $(0,1)$.

Proof of Lemma 1. Clearly $0\leqslant\xi_n\leqslant a\leqslant 1$ if x belongs to one of the disjoint intervals

$$\left(\prod_{j=1}^{n}\left(1+\frac{1}{q_j}\right),\ \prod_{j=1}^{n}\left(1+\frac{1}{q_j}\right)\left(1+\frac{a}{q_n^2-1}\right)\right)$$

where $q_1\geqslant 2$ and $q_{i+1}\geqslant q_i^2$ $(i=1,2,\ldots,n-1)$, and it can be seen by induction that the total length of these intervals $=a$. This proves Lemma 1.

It should be mentioned that a similar assertion holds for Sylvester's series (8), namely that if we put

$$R_n(x)=\frac{1}{Q_{n+1}}+\frac{1}{Q_{n+2}}+\ldots \tag{10}$$

then the random variables

$$\Xi_n=Q_n(Q_n-1)R_n(x) \tag{11}$$

are uniformly distributed in the interval $(0,1)$.

It follows by Lemma 1 that the random variables $\delta_n=\log(q_{n+1}/q_n^2)$ are asymptotically exponentially distributed for $n\to\infty$ with mean value 1, exactly as the random variables $\Delta_n=\log(Q_{n+1}/Q_n^2)$ in the case of Sylvester's series. Moreover the random variables δ_n are almost independent in the same sense as the random variables Δ_n. It can also be shown easily that the sequence $q_n=q_n(x)$ $(n=1,2,\ldots)$ of random variables is a homogeneous Markov-chain (similarly to the sequence $Q_n=Q_n(x)$) with the transition probabilities (1)

$$\pi_{jk}=P(q_{n+1}=k\mid q_n=j)=\frac{j^2-1}{k(k-1)} \quad \text{for} \quad k\geqslant j^2. \tag{12}$$

The probability distribution of $q_1(x)$ is given by

$$P(q_1=k)=\frac{1}{k(k-1)} \qquad (k\geqslant 2). \tag{13}$$

(1) $P(A|B)$ denotes the conditional probability of the event A with respect to the condition B.

From (13) and (12) the probability distribution of q_n may be determined for any n. As a matter of fact, putting

$$P_n(k) = P(q_n(x) = k) \tag{14}$$

we have the recurrence relations

$$P_n(k) = \sum_{l^2 \leqslant k} P_{n-1}(l) \frac{(l^2-1)}{k(k-1)}. \tag{15}$$

Using the facts mentioned, the following two theorems can be proved, by the same method as that used in [5] to prove the corresponding results for Sylvester's series:

THEOREM 1. *For almost all x the limit*

$$\lim_{n\to\infty} (q_{n+1}(x))^{1/2^n} = l(x) \tag{16}$$

exists and is finite and greater than 2.

THEOREM 2. *We have*

$$\lim_{n\to\infty} P\left(\frac{\log \frac{q_{n+1}(x)}{q_1(x)\ldots q_n(x)} - n}{\sqrt{n}} < y\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-u^2/2} du \tag{17}$$

for any real y.

It is implied by Theorem 2 that $\sqrt[n]{q_{n+1}(x)/q_1(x)\ldots q_n(x)}$ tends in measure to e. Still more is true, namely

THEOREM 3. *For almost all x*

$$\lim_{n\to\infty} \sqrt[n]{\frac{q_{n+1}(x)}{q_1(x)q_2(x)\ldots q_n(x)}} = e. \tag{18}$$

Theorem 3 can also be expressed by saying that *the strong law of large numbers is valid for the random variables δ_n*. As a matter of fact the assertion of Theorem 3 is equivalent to the statement that for almost all x we have

$$\lim_{n\to\infty} \frac{\delta_1(x)+\ldots+\delta_n(x)}{n} = 1. \tag{19}$$

Theorem 3 can be deduced from a theorem of Koksma and Salem ([2]).

([2]) See [12], p. 89, lemma. This lemma is a particular case of a result of I. Gál and J. F. Koksma [6].

To apply this theorem we need only to estimate the mean value of $\delta_n(x)\delta_{n+k}(x)$. Since the joint distribution of the variables $\delta_n(x)$ and $\delta_{n+k}(x)$ can be determined exactly from the formulae (12)-(15), this is possible. The corresponding result for Sylvester's series can be proved in the same way. Details will be published elsewhere.

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