vergent with denominator q_k satisfying

$$N < q_k < cN \quad (c < c_0, N < N_0)$$

has a measure greater than $1-\varepsilon$.

We shall return to this subject elsewhere.

REFERENCES

[1] P. Szüsz, *Bemerkungen zur Approximation einer reellen Zahl durch Brüche*, Acta Mathematica Academiae Scientiarum Hungaricae 6 (1955), p. 203-212.

[2] P. Turán, *Eine neue Methode in der Analysis und deren Anwendungen*, Budapest 1953 (for an enlarged version see the Chinese edition, Peking 1956).

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ON THE APPROXIMATE SOLUTIONS OF FUNCTIONAL EQUATIONS IN L^p SPACES

BY

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In papers [1] and [2] we have suggested an iterative method for solving non-linear functional equations in Banach spaces. This method may also be regarded as a generalization of Newton's well-known classical method. But this generalization is essentially different from that given by L. V. Kantorovitch [5].

The present paper contains a specification for the case of the real L^p-spaces and a real Hilbert space of the iterative method defined in paper [2]. An application to approximate solutions of operator equations in this space is also given. In particular we consider in the L^p-spaces an analogue of the method of steepest descent for non-linear operator equations.

The iterative process for solving non-linear functional equations is defined in papers [1], [2] as follows:

Let X be a Banach space and let $F(x)$, $x \in X$, be a non-linear continuous functional which is differentiable in the sense of Fréchet. Then the approximate process for solving the non-linear functional equation

$$F(x) = 0 \tag{1}$$

is defined by the formula

$$x_1 = x_0 - \frac{F(x_0)}{f_0(y_0)} y_0, \qquad x_{n+1} = x_n - \frac{F(x_n)}{f_n(y_n)} y_n, \tag{2}$$

where x_0 is the initial approximate solution, $f_n = F'(x_n)$ for $n = 0, 1, 2, \ldots$ denotes the Fréchet differential of $F(x)$ at the point $x = x_n$, and y_n are elements appropriately chosen in X, i. e. $\|y_n\| = 1$, $f_n(y_n) = \|f_n\|$, $n = 0, 1, 2, \ldots$, provided that such a choice is possible.

The specification for the case of the real L^p-spaces and the real Hilbert space consists in the appropriate choice of the elements y_n. It appears that in this case the choice of the elements y_n is effective and may be realized in a simple manner.

Now let X be a real Hilbert space and let $F(x)$ be a non-linear continuous functional defined on the closed sphere $S(x_0, r)$ of X with centre x_0 and radius r, having two continuous Fréchet derivatives.

Since the Fréchet differential $F'(x)$ of $F(x)$ is a linear functional on the Hilbert space X, it can be represented in the form

$$F'(x)y = (\Delta F(x), y), \quad y \in X,$$

and the element $\Delta F(x)$ is called the *gradient* of F.

The iterative process (2) assumes here the form

$$(3) \quad x_1 = x_0 - \frac{F(x_0)}{\|\Delta F(x_0)\|^2} \Delta F(x_0), \quad x_{n+1} = x_n - \frac{F(x_n)}{\|\Delta F(x_n)\|^2} \Delta F(x_n).$$

It is easy to see that such a choice of the elements $y_n = \Delta F(x_n)$ for $n = 0, 1, 2, \ldots$ satisfies the condition of Theorem 1 of [2].

A convergence theorem of process (3) to a solution of equation (1) is given by the general case considered in paper [2] (Theorem 1).

Now let X be the real space $L^p = L^p(a, b)$ with $p > 1$; then the conjugate space of L^p is L^q, where $1/p + 1/q = 1$.

Let $f \in L^q$ be an arbitrary linear functional on L^p. Putting

$$g(t) = \operatorname{sign} f(t) |f(t)|^{1/p-1} \quad (a \leqslant t \leqslant b),$$

we have

$$(4) \quad \int_a^b f(t) g(t) \, dt = \|f\|^q.$$

Thus, for $\bar{g} = g/\|g\|$ we obtain $\bar{g} \in L^p$, $\|g\| = 1$, and it is easy to see that the norm of the functional f is reached at the element $\bar{g} \in L^p$, *i. e.*

$$(5) \quad \int_a^b f(t) \bar{g}(t) \, dt = \|f\|.$$

As in the case of the Hilbert space, let us assume that $F(x)$ is a non-linear continuous functional defined on the closed sphere $S(x_0, r)$, having two Fréchet derivatives.

Since the Fréchet differential $F'(x)$ of $F(x)$ is a linear functional on L^p, we put

$$(6) \quad g(x)(t) = \operatorname{sign} F'(x)(t) |F'(x)(t)|^{1/(p-1)}$$

and obtain, by (4) and (5),

$$\int_a^b F'(x)(t) g(x)(t) \, dt = \|F'(x)\|^q, \quad \int_a^b F'(x)(t) \bar{g}(x)(t) \, dt = \|F'(x)\|,$$

where $\bar{g}(x) = g(x)/\|g(x)\|$.

Iterative process (3) assumes here the form

$$(7) \quad x_1 = x_0 - \frac{F(x_0)}{\|F'(x_0)\|^q} g(x_0), \quad x_{n+1} = x_n - \frac{F(x_n)}{\|F'(x_n)\|^q} g(x_n),$$

where $g(x_n)$ for $n = 0, 1, 2, \ldots$ is defined by formula (6).

It is easy to see that such a choice of the elements $y_n = g(x_n)$ satisfies the above mentioned condition of Theorem 1 of [2].

Thus, Theorem 1 of [2] gives also a convergence theorem of process (7) to a solution of equation (1).

Paper [3] contains an application of Newton's method of solving non-linear functional equations (1), defined by process (3), to the approximate solution of operator equations

$$(8) \quad P(x) = 0$$

in L^p-space. It is shown in [3] that equation (8) can be reduced to equation (1) by setting $F(x) = \|P(x)\|^p = 0$.

Thus we obtain in [3] the following approximate process for solving operator equation (8):

$$x_1 = x_0 - \frac{\|P(x_0)\|^p}{p\|f(x_0)\|^q} y_0, \quad x_{n+1} = x_n - \frac{\|P(x_n)\|^p}{p\|f(x_n)\|^q} y_n,$$

where

$$(9) \quad f(x) = \bar{P}'(x)(\operatorname{sign} P(x)(t)|P(x)(t)|^{p-1}) \in L^q,$$

$P'(x)$ denotes the Fréchet differential of $P(x)$, $\bar{P}'(x)$ is the adjoint of $P'(x)$ and

$$y_n = y_n(t) = \operatorname{sign} f(x_n)|f(x_n)(t)|^{1/p-1} \quad (n = 0, 1, 2, \ldots).$$

On the other hand, operator equation (8) can also be reduced to functional equation (1) by setting $F(x) = \|P(x)\| = 0$.

Applying process (7) to this case we obtain the following iterative process of solving operator equation (8):

$$(10) \quad x_1 = x_0 - \frac{\|P(x_0)\|^p}{\|f(x_0)\|^q} y_0, \quad x_{n+1} = x_n - \frac{\|P(x_n)\|^p}{\|f(x_n)\|^q} y_n,$$

where $f(x_n)$ and y_n have the same meaning as above.

Notice that we use here a result of S. Mazur [6] concerning the differentiability in Fréchet's sense of the norm $\|x\|$ in L^p to compute the Fréchet differential $F'(x)$ of $F(x) = \|P(x)\|$ provided that operator $P(x)$ is differentiable in Fréchet's sense.

It should also be remarked that in the case of a real Hilbert space, as it is shown in [4], process (10) may be regarded as an extension of the method of steepest descent.

We shall now give a convergence theorem of process (10) to a solution of operator equation (8).

Let $S(x_0, r)$ be a closed sphere in L^p with centre x_0 and radius r. Consider the non-linear operator equation (8), where P is a non-linear continuous operator defined on the sphere $S(x_0, r)$ with values in L^p with $1 < p$.

Let us assume that $P(x)$ is differentiable in the sense of Fréchet in the sphere $S(x_0, r)$.

We suppose also that there exists the Fréchet differential $f'(x)$ of $f(x)$, where $f(x)$ is defined by formula (9) and is bounded in the sphere $S(x_0, r)$, *i. e.*, that there exists a constant K such that

(11) $$\|f'(x)\| \leqslant K \quad \text{for every} \quad x \text{ of } S(x_0, r).$$

Sufficient conditions for the existence of a solution of equation (8) as well a convergence theorem of process (10) to this solution are given by the following

THEOREM 1. *Let us assume that the following conditions are satisfied:*

1° *the Fréchet differential $P'(x)$ of $P(x)$ exists and satisfies condition*

(12) $$\frac{1}{\|f(x_0)\|} \leqslant B_0;$$

2° *the Fréchet differential $f'(x)$ of $f(x)$ exists and satisfies condition* (11);

3° *the first approximate solution satisfies the inequality*

(13) $$\|x_1 - x_0\| = \frac{\|P(x_0)\|^p}{\|f(x_0)\|} \leqslant \eta_0,$$

where η_0 is a constant;

4° *the constants B_0, η_0 and K are subjected to the condition*

(14) $$pB_0K\eta_0 = h_0 \leqslant \tfrac{1}{2}$$

and

(15) $$r = \frac{1-\sqrt{1-2h_0}}{h_0}\eta_0.$$

Then equation (8) *has a solution x^* which belongs to the sphere $S(x_0, r)$ and the sequence of approximate solutions x_n defined by process* (10) *converges to x^*.*

For the error estimate we have the formula

(16) $$\|x_n - x^*\| \leqslant \frac{1}{2^{n-1}}(2h_0)^{2^n-1}\eta_0.$$

Proof. Multiplying equations (10) by $f(x_n)$ and integrating we obtain

$$\|P(x_n)\|^p = \int_a^b f(x_n)(t)(x_n - x_{n+1})(t)\,dt$$
$$= \int_a^b \operatorname{sign} P(x_n)(t)|P(x_n)(t)|^{p-1}P'(x_n)(x_n - x_{n+1})(t)\,dt.$$

Thus we have

(17) $$-\|P(x_n)\|^p = \int_a^b f(x_n)(t)(x_{n+1} - x_n)(t)\,dt.$$

Hence we get

$$\|P(x_n)\|^p = \|P(x_n)\|^p - \|P(x_{n-1})\|^p - \int_a^b f(x_{n-1})(t)(x_n - x_{n-1})(t)\,dt$$
$$= \|P(x_n)\|^p - \|P(x_{n-1})\|^p - p\int_a^b f(x_{n-1})(t)(x_n - x_{n-1})(t)\,dt +$$
$$+ (p-1)\int_a^b f(x_{n-1})(t)(x_n - x_{n-1})(t)\,dt.$$

Thus, we have by (17)

(18) $$\|P(x_n)\|^p \leqslant \|P(x_n)\|^p - \|P(x_{n-1})\|^p - p\int_a^b f(x_{n-1})(t)\cdot(x_n - x_{n-1})(t)\,dt.$$

Using the analogue of Taylor's formula we have by (18) and (11)

(19) $$\|P(x_n)\|^p \leqslant \frac{pK}{2}\|x_n - x_{n-1}\|^2.$$

Since $\|y_n\| = \|f(x_n)\|^{q/p}$, we have by (10)

(20) $$\|x_{n+1} - x_n\| = \frac{\|P(x_n)\|^p}{\|f(x_n)\|}.$$

Thus we obtain by (19) and (20)

(21) $$\|x_{n+1} - x_n\| \leqslant \frac{pK}{2\|f(x_n)\|}\|x_n - x_{n-1}\|^2.$$

We shall now estimate the norm $\|f(x_1)\|$:

$$\|f(x_1)\| \geqslant \|f(x_0)\| - \|f(x_0)-f(x_1)\| = \|f(x_0)\|\left(1 - \frac{\|f(x_0)-f(x_1)-f(x_1)\|}{\|f(x_0)\|}\right).$$

Hence, using the analogue of Lagrange's formula, we have by (12) and (11).

$$\|f(x_1)\| \geqslant \|f(x_0)\|\cdot(1-B_0K\eta_0) > \|f(x_0)\|\cdot(1-pB_0K\eta_0) = \|f(x_0)\|(1-h_0).$$

Thus, we obtain

$$\|f(x_1)\|^{-1} \leqslant \frac{1}{1-h_0}\cdot\frac{1}{\|f(x_0)\|} \leqslant \frac{B_0}{1-h_0} = B_1. \tag{22}$$

For $n=1$, (21) gives by (13) and (22)

$$\|x_2-x_1\| \leqslant \tfrac{1}{2}\cdot\frac{B_0}{1-h_0}pK\eta_0^2 = \tfrac{1}{2}\cdot\frac{h_0\eta_0}{1-h_0} = \eta_1. \tag{23}$$

Condition (14) is satisfied for $x = x_1$:

$$h_1 = pB_1K\eta_1 = p\cdot\frac{B_0}{1-h_0}\cdot\frac{K}{2}\cdot\frac{h_0\eta_0}{1-h_0} = \frac{h_0^2}{2(1-h_0)^2} \leqslant 2h_0^2 \leqslant \frac{1}{2}. \tag{24}$$

Thus, conditions 1°-4° are satisfied for $x = x_1$ replacing the numbers B_0, η_0 and h_0 by B_1, η_1 and h_1 defined by (22), (23) and (24). Hence we can define by induction the approximate solution x_n and the corresponding numbers B_n, η_n and h_n, which satisfy the formulas analogous to (13), (22), (23) and (24):

$$\|x_{n+1}-x_n\| \leqslant \eta_n, \tag{13'}$$

$$B_n = \frac{B_{n-1}}{1-h_{n-1}}, \tag{22'}$$

$$\eta_n = \frac{1}{2}\cdot\frac{h_{n-1}\eta_{n-1}}{1-h_{n-1}}, \tag{23'}$$

$$h_n = \frac{1}{2}\,\frac{h_{n-1}^2}{(1-h_{n-1})^2}. \tag{24'}$$

Using the same argument as in paper [2] we obtain from (13′), (22′), (23′) and (24′)

$$\|x_{n+p}-x_n\| \leqslant \frac{1}{2^{n-1}}(2h_0)^{2^n-1}\eta_0. \tag{25}$$

The same argument shows that all the approximate solutions x_n are contained in the sphere $S(x_0, r)$ defined by (15). It follows from (25) that the sequence (x_n) converges to an element x^* of the sphere $S(x_0, r)$. It follows from (19) that x^* is a solution of equation (8).

Formula (17) results from (25).

Remark 1. Condition (14) can be replaced by the following:

$$h_0 = pB_0^2KD \leqslant \tfrac{1}{2}, \tag{14'}$$

where D is a constant such that $\|P(x_0)\|^p \leqslant D$.

Remark 2. It is convenient to verify that condition (11) is satisfied in a constant sphere containing the sphere $S(x_0, r)$ where the radius r is defined by (15). For this purpose we can take, for instance, the sphere $S(x_0, r)$ where $r = 2\eta_0$ or $r = 2B_0D$.

If X is a real Hilbert space and P is a non-linear operator operating in this space and defined on the sphere $S(x_0, r)$, then the process (10) is of the form (see [3])

$$x_1 = x_0 - \frac{\|P(x_0)\|^2}{\|Q(x_0)\|^2}Q(x_0), \qquad x_{n+1} = x_n - \frac{\|P(x_n)\|^2}{\|Q(x_n)\|^2}Q(x_n), \tag{26}$$

where $Q(x) = P'(x)\overline{P}(x)$.

Suppose that there exists the Fréchet differential $Q'(x)$ of $Q(x)$ and that $Q'(x)$ is bounded in the sphere $S(x_0, r)$, *i. e.*, that there exists a constant K such that

$$\|Q'(x)\| \leqslant K \quad \text{for every} \quad x \text{ of } S(x_0, r). \tag{27}$$

As a particular case of Theorem 1 we have

THEOREM 2. *Let us assume that the following conditions are satisfied:*

1° *The Fréchet differential $P'(x)$ exists in the sphere $S(x_0, r)$ and satisfies the condition*

$$\frac{1}{\|Q(x_0)\|} \leqslant B_0; \tag{28}$$

2° *The Fréchet differential $Q'(x)$ of $Q(x)$ exists and satisfies the condition* (27);

3° *The first approximate solution satisfies the inequality*

$$\|x_1-x_0\| = \frac{\|P(x_0)\|^2}{\|Q(x_0)\|} \leqslant \eta_0; \tag{29}$$

4° *The constants B_0, η_0 and K are subjected to condition* (14) *with $p=2$ and r is defined by* (15).

Then equation (8) *has a solution which belongs to sphere $S(x_0, r)$ and the sequence of the approximate solutions x_n converges to x.*

For the error estimate we have the formula (16).

Remark 3. Remarks 1 and 2 are also applicable to Theorem 2.

REFERENCES

[1] M. Altman, *On the approximate solutions of non-linear functional equations*, Bulletin de l'Académie Polonaise des Sciences, Classe III, 5 (1957), p. 457-460.

[2] — *Concerning approximate solutions of non-linear functional equations*, ibidem 5 (1957), p. 461-465.

[3] — *On the approximate solutions of operator equations in L^p-spaces*, ibidem 5 (1957), p. 1099-1103.

[4] — *Connection between the method of steepest descent and Newton's method*, ibidem 5 (1957), p. 1031-1036.

[5] Л. В. Канторович, *О методе Ньютона*, Труды Математического Института им. Б. А. Стеклова 28 (1949), p. 101-144.

[6] S. Mazur, *Über schwache Konvergenz in den Räumen L^p*, Studia Mathematica 4 (1933), p. 128-133.

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ON CANTOR'S PRODUCTS

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G. Cantor [4] (see also [16], p. 122-127) considered the representation of a real number $x > 1$ in the form of the infinite product

$$x = \prod_{n=1}^{\infty}\left(1+\frac{1}{q_n}\right) \tag{1}$$

where $q_n = q_n(x)$ is a sequence of positive integers, which may be defined as follows: we choose for q_1 the least positive integer for which $1+1/q_1 \leqslant x$ and if $q_1, q_2, \ldots, q_{n-1}$ are already chosen, we choose for q_n the least positive integer for which $\prod_{k=1}^{n}(1+1/q_k) \leqslant x$. Clearly if x is contained in the interval $2^{k-1} < x \leqslant 2^k$ $(k = 1, 2 \ldots)$, then $q_1 = q_2 = \ldots = q_{k-1} = 1$, and $1 < x/\prod_{j=1}^{k-1}(1+1/q_j) \leqslant 2$. Thus we may restrict ourselves to the values of x lying in the interval $1 < x \leqslant 2$. In this case clearly

$$q_{n+1} \geqslant q_n^2 \quad (n = 1, 2, \ldots). \tag{2}$$

Let us put

$$E_0(x) = x, \quad E_n(x) = x\Big/\prod_{k=1}^{n}\left(1+\frac{1}{q_k}\right) \quad (n = 1, 2, \ldots). \tag{3}$$

It is easy to see that if x is rational, $x = a/b$ where a and b are positive integers, $b < a \leqslant 2b$, then we obtain by the algorithm described above a finite representation for x of the form

$$\frac{a}{b} = \prod_{n=1}^{N}\left(1+\frac{1}{q_n}\right) \tag{4}$$

since putting $E_n(a/b) = a_n/b_n$ we have $a_{n+1}-b_{n+1} < a_n-b_n$; it follows that $N \leqslant a-b$.