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THE EXISTENCE OF VECTOR FUNCTION SPACES WITH DUALS OF INTEGRAL TYPE

BY

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§ 1. Introduction. Let C be the field of complex numbers, E a locally compact topological space, and C^E the vector space of mappings of E into C. It is understood that a vector subspace of C^E is always "over" C. Let A be a vector space of C^E with semi-norm $\mathcal{N}^A$, with values $\mathcal{N}^A(x)$ for $x \in A$. We say that A is *C-proper* if $x \in A$ implies that $|x|$ and the conjugate $\bar{x}$ is in A. We term $\mathcal{N}^A$ *monotone* if the conditions x and y in A and $|x| \leqslant |y|$ imply that $\mathcal{N}^A(x) \leqslant \mathcal{N}^A(y)$. When $\mathcal{N}^A$ is monotone it is clear that $\mathcal{N}^A(x) = \mathcal{N}^A(|x|)$. We say that $\mathcal{N}^A$ and A are *trivial* if $\mathcal{N}^A(x) = 0$ for each $x \in A$.

Vector subspaces of C^E which satisfy Conditions I and II below are termed *MT-spaces*. They are said to have "duals of integral type" because of the satisfaction of Condition II.

Condition I. *Under condition* I, *A shall be a C-proper vector subspace of C^E with monotone semi-norm $\mathcal{N}^A$, and shall contain* $\mathcal{K}_C(E)$ (see [5], p. 48) *as an everywhere-dense subspace.*

We write $\mathcal{K}_C$ in place of $\mathcal{K}_C(E)$.

The measure dual $\mathscr{A}'$. Let A' be the dual of A, supposing that A satisfies I. Given $\eta \in A'$ we introduce the C-measure

$$\hat{\eta} = \eta | \mathcal{K}_C \quad \text{(cf. [7], p. 169).}$$

The transformation $\eta \to \hat{\eta}$ maps A' homomorphically onto a subspace $\mathscr{A}'$ of the space of C-measures. We term $\mathscr{A}'$ the *measure dual* of A. The map $\eta \to \hat{\eta}$ is an *isomorphism*, since $\hat{\eta} = 0$ implies that $\eta | \mathcal{K}_C = 0$, and this in turn implies that $\eta = 0$, since A is the closure of $\mathcal{K}_C$ in the topology defined by $\mathcal{N}^A$.

The set $\mathscr{A}^u$. We norm $\mathscr{A}'$ by setting

$$\|\alpha\|_{\mathscr{A}'} = \sup_v \left| \int v \, d\alpha \right| \quad (\alpha \in \mathscr{A}') \tag{1.1}$$

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taking the sup over $v \in \mathcal{K}_C$ with $\mathcal{N}^A(v) \leqslant 1$. The integral is the Bourbaki integral as exposed in [7]. An $a = \mathcal{A}'$, with norm at most 1, will be termed *subunit*. The subset of all subunits $a \in \mathcal{A}'$ will be denoted by $\mathcal{A}^u$. From Theorem 10.1 of [7] we infer the following

LEMMA 1.1. *If a is a C-linear form on $\mathcal{K}_C$ a necessary and sufficient condition that a be a C-measure in $\mathcal{A}^u$ is that for each $v \in \mathcal{K}_C$*

(1.2) $$|a(v)| \leqslant \mathcal{N}^A(v).$$

CONDITION II. *Condition* II *presupposes that A satisfy* I, *and requires further that*

(a) *each $x \in A$ shall be a-integrable for each $a \in \mathcal{A}'$,*

(b) *for each $x \in A$ and $\eta \in A'$*

(1.3) $$\eta(x) = \int x\, d\eta.$$

MT-spaces include the spaces $\mathcal{L}^p_C(E)$, $p \geqslant 1$, as we shall show in § 2, and the Orlicz spaces suitably defined. They also include much more general vector subspaces of $\boldsymbol{C}^E$ in accordance with the following theorem, to be proved in § 2:

A necessary and sufficient condition that a vector subspace of $\boldsymbol{C}^E$ which satisfies Condition I *be an* MT-*space is that for some positive constant m_a*

(1.4) $$N^1_C(x, a) \leqslant m_a \mathcal{N}^A(x)$$

for each $x \in A$ and C-measure $a \in \mathcal{A}'$.

For the definition of $N^1_C(x, c)$ see [7], p. 155, and [5], p. 127. A special consequence of this theorem is that when E is a discrete topological space, an arbitrary vector subspace A of $\boldsymbol{C}^E$ which satisfies Condition I is always an MT-space. See § 2. As another consequence of the above theorem we shall derive, in § 2, a Condition III on A that implies (1.4) but does not itself involve the measure dual $\mathcal{A}'$.

We shall establish the existence of MT-spaces by three kinds of processes as follows:

(Ω_1) *Logical characterization of* MT-*spaces*. The preceding condition (1.4) comes under this category. See § 2.

(Ω_2) *A priori prescription of a set ω of subunit $\boldsymbol{C}$-measures*. The set ω is conditioned so as to serve as the set of subunit C-measures in the measure dual of a maximal MT-space A_ω. "Saturated" sets ω are defined, and it is proved that the mapping $\omega \to A_\omega$ of the saturated sets ω for which the *real envelope*

$$\varrho_\omega(v) = \sup_{a \in \omega} \int |v|\, d|a| \qquad (v \in \mathcal{K}_C)$$

is finite valued, is *onto* the ensemble of maximal MT-spaces and is one-to-one. See § 3.

(Ω_3) *Generation of* MT-*spaces by intersections and unions of* MT-*spaces*. Given a collection (A_i), $i \in I$, of MT-subspaces of $\boldsymbol{C}^E$ we introduce the spaces

$$\bigcap_i A_i = J, \qquad \bigcup_i A_i = V \qquad (i \in I)$$

and assign a semi-norm to J, and when possible to V, so as to define MT-spaces J and V respectively. The fundamental instrument here is the lattice Λ. See § 4.

The lattice Λ. As shown by Bourbaki in [5] the space $\mathcal{M}(E)$ of measures on E is a partially ordered vector space such that each non-empty majorized part of $\mathcal{M}(E)$ admits a least upper bound in $\mathcal{M}(E)$. To motivate our generalization observe that the ordering of measures can be made to depend upon the ordering of positive measures μ, restricted to their values on $\mathcal{K}_+$. If μ is a measure, then for $f \in \mathcal{K}_+$

$$\mu(f) = N_1(f, \mu).$$

Thus the ordering of measures may be replaced by an equivalent ordering of the semi-norms $N_1(\cdot, \mu)$. We generalize by introducing the lattice Λ of *all* semi-norms on $\mathcal{K}_C$.

Note. The above definition of Condition I omits the condition, found in [7] and [9], that A be *non-trivial*. The definition of MT-spaces is thereby affected. These spaces now include those trivial spaces A which satisfy the remaining defining conditions. We have found it desirable to admit these trivial spaces in order to presently develop the theory of spaces locally of MT-type. For it turns out that an MT-space A, which is not itself trivial, may have a section $A|K$ by a compact set K, such that $A|K$ is trivial. The theorems formally stated in [7] and [9] are obviously valid if trivial MT-spaces are admitted.

§ 2. First theorems on MT-spaces. In § 11 of [7] we have shown that for each x in an MT-space A with semi-norm $\mathcal{N}^A$

(2.0) $$\mathcal{N}^A(x) = \sup_{a \in \mathcal{A}^u} \int |x|\, d|a|.$$

The semi-norm so given has been extended in [7] as a mapping of $\boldsymbol{C}^E$ into $\bar{\boldsymbol{R}}$ (the positive real axis with $+\infty$ adjoined) by the formula

(2.1) $$\mathcal{N}^A(y) = \sup_{a \in \mathcal{A}^u} \int^* |y|\, d|a| \qquad (y \in \boldsymbol{C}^E)$$

where use is made of the superior integral of Bourbaki. The absolute measure $|\alpha|$ is defined in Theorem 3.1 of [7]. Another extension is as follows. Let h be an arbitrary mapping of E into $\bar{R}_+$. We define $\mathcal{N}^A(h)$ by (2.1), with y replaced by h.

The subspace $\mathcal{F}^A$ *of* C^E. The vector subspace of C^E on which $\mathcal{N}^A(y) < \infty$ has been denoted by $\mathcal{F}^A$. The space $\mathcal{F}^A$ is a C-proper vector subspace of C^E on which $\mathcal{N}^A$, as extended, is a monotone semi-norm. We term $\mathcal{N}^A$, as extended, the *natural extension* of $\mathcal{N}^A|A$. The measure dual $\mathcal{A}'$, with its norm and $\mathcal{A}^u$, are completely determined by the restriction $\mathcal{N}^A|\mathcal{K}_C$.

Carrier of B. When B is a semi-normed vector subspace of C^E, the subspace of C^E underlying B, taken apart from its semi-norm, will be called the *carrier* of B.

The MT-*space* A_0. Given a vector subspace A of C^E, satisfying Condition I, let A_0 be the vector space with carrier that of $\mathcal{K}_C$, and semi-norm $\mathcal{N}^{A_0}$ induced by $\mathcal{N}^A$. The space A_0 clearly satisfies Condition I. Condition II is automatically satisfied since each $v \in A_0$ is α-integrable for arbitrary C-measure α, and since for $\eta \in \mathcal{A}'_0$

$$\eta(v) = \hat{\eta}(v) = \int v d\hat{\eta} \qquad (v \in \mathcal{K}_C) \tag{2.2}$$

merely as a matter of notation. One thus has

$$\mathcal{N}^A|A_0 = \mathcal{N}^{A_0}, \qquad \mathcal{A}' = \mathcal{A}'_0, \qquad \mathcal{A}^u = \mathcal{A}^u_0. \tag{2.3}$$

When A is an MT-space, the natural extensions of $\mathcal{N}^A$ and of $\mathcal{N}^{A_0}$ are *both* defined, and are identical. In this case

$$\mathcal{F}^A = \mathcal{F}^{A_0}. \tag{2.4}$$

The measure dual $\mathcal{A}'$ is defined when A merely satisfies Condition I, However, under Condition I, (2.1) does not necessarily hold on A, as will follow from Corollary 2.2. We can thus speak of (2.1) as giving the "natural extension" of $\mathcal{N}^A$ *only when* A satisfies Condition II as well as I, and refer to $\mathcal{F}^A$ only in that case. The natural extension of $\mathcal{N}^{A_0}$ is *always* well-defined, if Condition I is satisfied by A.

Maximal MT-*spaces*. An MT-space A is termed *maximal* if A is a proper vector subspace of no MT-space B with semi-norm of A induced by that of B. With this understood the following theorem is a consequence of Corollary 12.2 and Theorem 13.1 of [7]:

THEOREM 2.1. *When* A *is an* MT-*space,* $\mathcal{F}^A$ *is complete and the closure* $\bar{A}$ *of* A *in* $\mathcal{F}^A$ *is a complete and maximal* MT-*space. Conversely, if* A *is a maximal* MT-*subspace of* C^E, *then* $A = \bar{A}$ *in* $\mathcal{F}^A$.

In terms of A and its MT-subspace A_0 with carrier $\mathcal{K}_C$, we have the following corollary:

COROLLARY 2.1. *If* A *is a maximal* MT-*subspace of* C^E *then* $\bar{A}_0 = A$, *where* $\bar{A}_0$ *is the closure of* A_0 *in* $\mathcal{F}^A$.

Proof. We have already seen in (2.4) that $\mathcal{F}^A = \mathcal{F}^{A_0}$. Since A_0 is everywhere dense in A in the topology of A and hence of $\mathcal{F}^A$, $A \subset \bar{A}_0$. Now A is a maximal MT-subspace of $\mathcal{F}^A$ by hypothesis, while $\bar{A}_0$ is an MT-subspace of $\mathcal{F}^{A_0} = \mathcal{F}^A$ by Theorem 2.1. Hence the inclusion $A \subset \bar{A}_0$ implies the equality $A = \bar{A}_0$.

We come to a new theorem:

THEOREM 2.2. *If* A *satisfies Condition* I, *a necessary and sufficient condition that* A *be an* MT-*space is that for each* $\alpha \in A'$ *there exists a constant* $m_\alpha \geqslant 0$ *such that*

$$N^1_C(x, \alpha) \leqslant m_\alpha \mathcal{N}^A(x) \qquad (x \in A). \tag{2.5}$$

When (2.5) *holds the minimum choice of* m_α *is* $\|\alpha\|_{\mathcal{A}'}$.

When A is an MT-space, (2.5) holds with $m_\alpha = \|\alpha\|_{\mathcal{A}'}$ in accordance with (2.0), since $\mathcal{N}^1_C(x, \alpha) = \int |x| d|\alpha|$.

When (2.5) holds $\|\alpha\|_{\mathcal{A}'}$ is a minimum choice of m_α, since, with $v \in \mathcal{K}_C(E)$ and $\mathcal{N}^A(v) \leqslant 1$, $\|\alpha\|_{\mathcal{A}'} = \sup|\alpha(v)| \leqslant \sup\int |v| d|\alpha| \leqslant m_\alpha$ using (2.5).

It remains to prove that (2.5) is sufficient.

Given $x \in A$ and $\alpha \in \mathcal{A}'$ we shall first prove that x is α-integrable. By Condition I, x is in the closure of $\mathcal{K}_C$ in A, in the topology defined in A by $\mathcal{N}^A$. It follows from (2.5) that x is in the closure of $\mathcal{K}_C(E)$ in $\mathcal{L}^1_C(\alpha)$, in the topology defined by $\mathcal{N}^1_C(\alpha)$. Hence x is α-integrable. See [7], § 4.

We conclude by showing that for $\eta \in A'$ and $x \in A$, (1.3) holds. Now (1.3) holds for $x \in \mathcal{K}_C$ by definition of $\hat{\eta}$. Moreover both members of (1.3) are continuous in the topology T_1 defined on A by $\mathcal{N}^A$. The left member η of (1.3) is so continuous since η is in A'. The right member $\int x d\hat{\eta}$ is so continuous since $\int x d\hat{\eta}$ is continuous on A in the topology T_2 defined by $\mathcal{N}^1_C(\cdot, \hat{\eta})$, and since T_1 is finer than T_2 by (2.5). Since $\mathcal{K}_C$ is everywhere dense in A in the topology T_1, (1.3) holds.

COROLLARY 2.2. *If* A *satisfies Condition* I *a necessary and sufficient condition that* A *be an* MT-*space is that for each* $x \in A$, *and* $\alpha \in A'$,

$$\mathcal{N}^A(x) = \sup_{\alpha \in \mathcal{A}^u} \int^* |x| d|\alpha|. \tag{2.6}$$

The condition (2.6) is necessary in accordance with (2.0). If the condition (2.6) is satisfied, then for each $x \in A$ and $\alpha \in \mathcal{A}'$

$$N^1_C(x, \alpha) \leqslant \|\alpha\|_{\mathcal{A}'} \mathcal{N}^A(x). \tag{2.7}$$

That A is an MT-space now follows from Theorem 2.2.

The filtering sets ${}_-\varphi^E_p$ *and* ${}_+\varphi^E_h$. To proceed earlier notation must be recalled. With Bourbaki let $\mathcal{G}_+$ denote the set of lower semi-continuous mappings of E into $\bar{\boldsymbol{R}}_+$. For $p \in \mathcal{G}_+$ let ${}_-\varphi_p^E$ denote the ensemble of mappings $f \in \mathcal{K}_+(E)$ with $f \leqslant p$, "filtering" for the relation $\leqslant$ (see [2], p. 35). Let h be given in $\bar{\boldsymbol{R}}_+^E$. Let ${}_+\varphi_h^E$ denote the ensemble of mappings $p \in \mathcal{G}_+$ with $p \geqslant h$, filtering for the relation $\geqslant$ (cf. also [7], § 6). When no ambiguity arises we shall write ${}_-\varphi_p$ and ${}_+\varphi_h$ in place of ${}_-\varphi_p^E$ and ${}_+\varphi_h^E$.

Properties of A_0. Given a vector subspace A of C^E satisfying Condition I, A_0 has been previously defined in § 2. In terms of the natural extension of $\mathcal{N}^{A_0}$ we shall establish the relation

$$\mathcal{N}^{A_0}(p) = \sup_{f \in {}_-\varphi_p} \mathcal{N}^A(f) \qquad (p = \mathcal{G}_+). \tag{2.8}$$

By definition of the natural extension of $\mathcal{N}^{A_0}$

$$\mathcal{N}^{A_0}(p) = \sup_{\alpha \in \mathcal{A}^u} \int^* p\, d|\alpha| = \sup_{\alpha \in \mathcal{A}^u} \sup_{f \in {}_-\varphi_p} \int f d|\alpha|,$$

making use of the definition of the superior integral of p. Continuing

$$\mathcal{N}^{A_0}(p) = \sup_{f \in {}_-\varphi_p} \sup_{\alpha \in \mathcal{A}^u} \int f d|\alpha| = \sup_{f \in {}_-\varphi_p} \mathcal{N}^A(f),$$

thus establishing (2.8).

In terms of A_0 Corollary 2.2 may be stated as follows. If A satisfies Condition I, a necessary and sufficient condition that A be an MT-space is that for each $x \in A$, $\mathcal{N}^{A_0}(x) = \mathcal{N}^A(x)$.

We are led to Condition III:

CONDITION III. *Under this condition A shall satisfy Condition* I *and be such that for each real $h \geqslant 0$ in A there exist a sequence (p_n) of $p_n \in {}_+\varphi_h$ such that (in terms of the extended semi-norm $\mathcal{N}^{A_0}$)*

$$\inf_n \mathcal{N}^{A_0}(p_n) \leqslant \mathcal{N}^A(h). \tag{2.9}$$

THEOREM 2.3. *A sufficient condition that a vector subspace A of C^E be an* MT-*space is that A satisfy Condition* III.

We show that A is an MT-space by showing that (2.5) holds in the form

$$\int^* |x|\, d|\alpha| \leqslant \|\alpha\|_{\mathcal{A}'} \mathcal{N}^A(x) \quad \text{for each} \quad x \in A. \tag{2.10}$$

In accordance with formula (2.1) the semi-norm $\mathcal{N}^{A_0}$ extending $\mathcal{N}^A | K_C$ over C^E is such that

$$\int^* p_n d|\alpha| \leqslant \|\alpha\|_{\mathcal{A}'} \mathcal{N}^{A_0}(p_n). \tag{2.11}$$

Making use of the monotonicity of the superior integral, of (2.11) and of (2.9), respectively,

$$\int^* h d|\alpha| \leqslant \inf_n \int^* p_n d|\alpha| \tag{2.12}$$

$$\leqslant \|\alpha\|_{\mathcal{A}'} \inf_n \mathcal{N}^{A_0}(p_n) \leqslant \|\alpha\|_{\mathcal{A}'} \mathcal{N}^A(h) \qquad (0 \leqslant h \in A).$$

On setting $h = |x|$ in (2.12), one obtains (2.10) and (2.5), thereby establishing the theorem.

COROLLARY 2.3. *If A satisfies Condition* I *a sufficient condition that A be of* MT-*type is that for each $h \geqslant 0$ in A there exists a sequence (p_n) of $p_n \in {}_+\varphi_h \cap A$ such that*

$$\inf_n \mathcal{N}^A(p_n) = \mathcal{N}^A(h). \tag{2.13}$$

Since p_n is in $\mathcal{G}_+$, it follows from (2.8) that $\mathcal{N}^{A_0}(p_n) \leqslant \mathcal{N}^A(p_n)$. We see that (2.13) implies (2.9). Corollary 2.3 then follows from Theorem 2.3.

COROLLARY 2.4. *If E is a discrete topological space any vector subspace A of $\boldsymbol{C}^E$ which satisfies Condition* I *is an* MT-*space.*

When E is discrete, each $x \in \boldsymbol{C}^E$ is continuous. Condition (2.13) is satisfied, taking $p_h = h$ for each n.

The spaces $B = \mathcal{L}_C^p(\beta)$. Let β be a $\boldsymbol{C}$-measure on E. With each mapping $x \in \boldsymbol{C}^E$ let there be associated a number

$$\mathcal{N}_C^p(x, \beta) = \left[\int^* |x|^p d|\beta|\right]^{1/p} \qquad (p \geqslant 1) \tag{2.14}$$

finite or infinite. As in [5], p. 131, let $\mathcal{F}_C^p(\beta)$ be the vector space over $\boldsymbol{C}$ of mappings $x \in \boldsymbol{C}^E$ for which $\mathcal{N}_C^p(x, \beta)$ is finite, with a semi-norm defined by (2.14). Let $B = \mathcal{L}_C^p(\beta)$ be the closure of $\mathcal{K}_C(E)$ in $\mathcal{F}_C^p(\beta)$. The space B is a vector space over $\boldsymbol{C}$ with a semi-norm defined by (2.14). Such a semi-norm is monotone on B. We need the following:

(i) *The application $x \to |x|$ maps the space $\mathcal{L}_C^p(\beta)$ uniformly continuously onto the space $\mathcal{L}^p(|\beta|)$, as defined by Bourbaki. The latter space is a subspace of $\mathcal{L}_C^p(\beta)$.*

The proof of (i) is identical with the proof of Lemma 4.1 in [7], p. 156, for the case $p = 1$, replacing the subscript and superscript 1 by p. As a consequence of (i), $|x|$ is in B with x. Making use of the fact that $\mathcal{N}^B(x) = \mathcal{N}^B(\bar{x})$, and the definition of B as a closure of $\mathcal{K}_C(E)$, we see that $\bar{x}$ is in B with x. Thus B is a $\boldsymbol{C}$-proper vector space.

LEMMA 2.1. *The space $B = \mathcal{L}_C^p(\beta)$ is an* MT-*space for each C-measure* β.

We shall establish Lemma 2.1 using Corollary 2.3. We note first that B is C-proper with a monotone semi-norm. Moreover $\mathcal{K}_C$ is in B and everywhere dense in B.

To show that B is an MT-space it is now sufficient to show that (2.13) is satisfied. This will follow if we show that for $h \geqslant 0$ in B

$$\inf_{q \in +\varphi_h} \int^* q^p \, d|\beta| = \int h^p \, d|\beta|. \tag{2.15}$$

In accordance with Theorem 3, [5], p. 151, given $\varepsilon > 0$, there exists a β-integrable $k \in \mathcal{I}_+(E)$ such that $k \geqslant h^p$, and

$$\int (k - h^p) \, d|\beta| < \varepsilon.$$

Choose $r \geqslant 0$ in $\boldsymbol{R}_+^E$ so that $r^p = k$. Then $r^p \geqslant h^p$, r is in $_+\varphi_h$ and

$$0 \leqslant \int r^p \, d|\beta| - \int h^p \, d|\beta| < \varepsilon.$$

Relation (2.15) follows and the lemma is proved.

§ 3. The a priori prescription of the measure dual of a maximal MT-space. Given E we shall say that an MT-space A is *induced* by an MT-space B, if A is a vector subspace of B with a semi-norm induced by that of B. We then term A an MT-*subspace* of B. To prescribe the measure dual of A is to prescribe the measure dual of $\bar{A}$, the maximal MT-space which induces A (cf. Theorem 2.1). The measure dual of A does not uniquely determine A in general, since there may be many MT-spaces induced by $\bar{A}$, all with the same measure dual. It is for this reason that we turn to the problem of the a priori prescription of the measure dual $\mathcal{A}'$ of a *maximal* MT-space A.

To prescribe a measure dual $\mathcal{A}'$ (assumed normed) is equivalent to prescribing the subspace $\mathcal{A}^u$ of subunit C-measures in $\mathcal{A}'$, and this is our immediate objective. To describe the essential characteristics of $\mathcal{A}^u$ we begin with two definitions.

Definition of ϱ_ω. Let ω be an arbitrary non-empty ensemble of C-measures α on E. Let $|\omega|$ denote the ensemble of measures $|\alpha|$ as α ranges over ω. The mapping ϱ_ω of $\mathcal{K}_C$ into $\bar{\boldsymbol{R}}_+$ with values

$$\varrho_\omega(v) = \sup_{\alpha \in \omega} \int |v| \, d|\alpha| \qquad (v \in \mathcal{K}_C) \tag{3.1}$$

will be termed the *env. sup.* of $|\omega|$. It is clear that $\varrho_\omega(u) \leqslant \varrho_\omega(v)$ if $|u| \leqslant |v|$, and that

$$\varrho_\omega(u+v) \leqslant \varrho_\omega(u) + \varrho_\omega(v), \qquad \varrho_\omega(\lambda u) = |\lambda| \, \varrho_\omega(u) \qquad (0 \neq \lambda \in C).$$

The sets ω^*. Given the set of C-measures ω, let ω^* denote the set of C-measures *saturating* ω, that is the set of C-measures β such that

$$|\beta(v)| \leqslant \varrho_\omega(v) \qquad (v \in \mathcal{K}_C). \tag{3.2}$$

It is clear that $\omega \subset \omega^*$. A set ω such that $\omega = \omega^*$ will be said to be *saturated.*

We state a fundamental theorem.

THEOREM 3.1 (a). *Given an MT-space A set $\mathcal{A}^u = \omega$. Then ϱ_ω is finite valued and ω is saturated.*

(b). *Conversely if ω is an arbitrary non-empty set of C-measures such that ϱ_ω is finite-valued, there exists a unique maximal MT-space A_ω such that $\mathcal{A}_\omega^u = \omega^*$. For such a space*

$$\mathcal{N}^{A_\omega} | \mathcal{K}_C = \varrho_\omega. \tag{3.3}$$

Proof of (a). By hypothesis A is an MT-space. Hence for $v \in \mathcal{K}_C$ and $\omega = \mathcal{A}^u$, $\mathcal{N}^A(v) = \varrho_\omega(v)$ in accordance with the definition (3.1) and formula (2.0). Thus ϱ_ω is finite-valued. Each C-measure β such that

$$|\beta(v)| \leqslant \mathcal{N}^A(v) = \varrho_\omega(v) \qquad (v \in \mathcal{K}_C) \tag{3.4}$$

is in $\mathcal{A}^u = \omega$ by Lemma 1.1, so that ω is saturated.

Proof of (b). *The space* B_ω. Let the carrier of $\mathcal{K}_C$, with semi-norm $v \to \varrho_\omega(v)$, be denoted by B_ω. The values $\varrho_\omega(v)$ are finite by hypothesis, and as a semi-norm, ϱ_ω is monotone. Conditions I are satisfied, and Conditions II trivially. Thus B_ω is an MT-space. We write $\mathcal{B}_\omega^u$ in place of $(\mathcal{B}_\omega)^u$.

We shall prove that $\mathcal{B}_\omega^u = \omega^*$. If $\beta \in \mathcal{B}_\omega^u$, $|\beta(v)| \leqslant \varrho_\omega(v)$ for $v \in B_\omega$, by virtue of Lemma 1.1, so that (3.2) holds. Thus $\mathcal{B}_\omega^u \subset \omega^*$. Conversely if $\beta \in \omega^*$

$$|\beta(v)| \leqslant \varrho_\omega(v) = \mathcal{N}^{B_\omega}(v) \qquad (v \in \mathcal{B}_\omega) \tag{3.5}$$

by definition of ω^* and of $\mathcal{N}^{B_\omega}$. Hence β is in $\mathcal{B}_\omega^u$ by Lemma 1.1. We conclude that $\mathcal{B}_\omega^u = \omega^*$.

The uniqueness of A_ω. Any maximal MT-space H such that $\mathcal{H}^u = \omega^*$ must be identical with $\bar{B}_\omega$. For the carrier of H includes the carrier of $\mathcal{K}_C$ as a vector subspace, and by (2.0) has a semi-norm such that for $v \in \mathcal{K}_C$

$$\mathcal{N}^H(v) = \sup_{\alpha \in \omega^*} \int |v| \, d|\alpha| = \varrho_\omega(v). \tag{3.6}$$

In the terminology of Corollary 2.1, $H_0 = B_\omega$, and it follows from this corollary that $H = \bar{B}_\omega$. Thus H is uniquely determined.

This completes the proof of the theorem.

We now explore the possibility of the a priori prescription of the normed dual of an MT-space. We have seen in Theorem 3.1 how to prescribe the ensemble ω^* of subunit C-measures of a maximal MT-space A. The space $\mathcal{A}'$ is thereby uniquely determined as the set of C-multiples of C-measures in ω^*. Moreover a norm $a \to \|a\|_{\mathcal{A}'}$ is also uniquely determined. Conversely we state the following:

THEOREM 3.2. *Let $\mathcal{S}$ be a non-empty vector space of C-measures a on E with norm $a \to \|a\|_{\mathcal{S}}$. Let ω be the set of subunit C-measures in $\mathcal{S}$. If ϱ_ω is finite-valued and if $\omega = \omega^*$ there exists a unique maximal MT-space A (namely A_ω) such that $\mathcal{A}' = \mathcal{S}$ and*

$$\|a\|_{\mathcal{A}'} = \|a\|_{\mathcal{S}} \quad (a \in \mathcal{A}'). \tag{3.7}$$

If A exists it is unique. For the condition $\mathcal{A}' = \mathcal{S}$ implies that $\mathcal{A}^u = \omega^*$, so that $A = A_\omega$ by Theorem 3.1 (b). The MT-space A_ω of Theorem 3.1 actually has the property that $\mathcal{A}'_\omega = \mathcal{S}$, since $\mathcal{A}^u_\omega = \omega^*$ by Theorem 3.1 (b). It remains to show that (3.7) is satisfied by $A = A_\omega$.

An $a \in \mathcal{A}'$ which failed to satisfy (3.7) could not vanish. Without loss of generality we could then suppose that $\|a\|_{\mathcal{A}'} = 1$, $\|a\|_{\mathcal{S}} = r \neq 1$.

Our choice of $A = A_\omega$ was such that $\mathcal{A}^u = \omega^*$, so that $a \in \mathcal{A}^u = \omega^* = \omega$. As an element in ω a is subunit in $\mathcal{S}$. Hence $r < 1$. We then have $\|a/r\|_{\mathcal{A}'} = 1/r > 1$, $\|a/r\|_{\mathcal{S}} = 1$. Thus a/r is in ω but not in $\mathcal{A}^u$. From this contradiction we infer that (3.7) holds.

This establishes the theorem.

The mapping $\omega \to A_\omega$ defined in Theorem 3.1, if restricted to saturated sets, is one-to-one. We make this more explicit.

The ensemble Ω. Let Ω be the ensemble of non-empty sets ω of C-measures on E for which ϱ_ω is finite-valued and ω saturated.

The ensemble Θ. Let Θ be the ensemble of maximal MT-subspaces of C^E.

Theorem 3.1 has the corollary.

COROLLARY 3.1. *The mapping $\omega \to A_\omega$ of Ω into Θ in which $\omega \in \Omega$ corresponds to $A_\omega \in \Theta$ if and only if $\mathcal{A}^u_\omega = \omega$, is a one-to-one mapping of Ω onto Θ.*

This corollary suggests a partial ordering of Ω and Θ. We prepare for this by the following theorem:

THEOREM 3.3. *If A and B are two MT-spaces; then*

(a) *the following three conditions are equivalent:*

$$\mathcal{N}^A(v) \leqslant \mathcal{N}^B(v) \quad (v \in \mathcal{K}_C), \tag{3.8$'$}$$

$$\mathcal{A}^u \subset \mathcal{B}^u, \tag{3.8$''$}$$

$$\mathcal{N}^A(x) \leqslant \mathcal{N}^B(x) \quad (x \in C^E), \tag{3.8$'''$}$$

where $\mathcal{N}^A$ and $\mathcal{N}^B$ in (3.8)$'''$ are the natural extensions over C^E of $\mathcal{N}^A$ and $\mathcal{N}^B$, as given a priori on A and B;

(b) *if any one of the three relations (3.8) is a strict equality, the remaining two relations are likewise;*

(c) *the relations (3.8) imply the relation $\mathcal{F}^A \supset \mathcal{F}^B$ and, if A is maximal, that $A \supset B$.*

Relation (3.8)$'$ implies (3.8)$''$. For the condition that a C-measure a be in $\mathcal{A}^u$, by Lemma 1.1, is that $|a(v)| \leqslant \mathcal{N}^A(v)$ for each $v \in \mathcal{K}_C$. Hence (3.8$'$) implies that $|a(v)| \leqslant \mathcal{N}^B(v)$, or by Lemma 1.1 that $a \in \mathcal{B}^u$.

Relation (3.8)$''$ implies (3.8)$'''$. This follows from the formulas giving the natural extensions of $\mathcal{N}^A$ and $\mathcal{N}^B$.

Relation (3.8)$'''$ implies (3.8)$'$. This is a consequence of the fact that the extension formulas are valid if restricted to $\mathcal{K}_C$.

Proof of (b). Strict equality in any one of the relations (3.8) is equivalent to inequality or inclusion in both senses, and so implies strict equality in all relations (3.8).

Proof of (c). When (3.8)$'''$ holds, $\mathcal{F}^A \supset \mathcal{F}^B$, and the topology of $\mathcal{F}^B$ is finer than the topology of $\mathcal{F}^A$. Hence the closure $\bar{B}$ of $\mathcal{K}_C$ in the topology of $\mathcal{F}^B$ is contained in the closure $A = \bar{A}$ of $\mathcal{K}_C$ in the topology of $\mathcal{F}^A$. Thus (cf. [2], p. 21)

$$A \supset \bar{B} \supset B. \tag{3.9}$$

Note. It would be an error to affirm that when A and B are maximal MT-subspaces of C^E the relation $A \supset B$ implies the relations (3.8). Simple examples on a discrete topological space C^E show that when $A \supset B$, neither the relation $\mathcal{N}^A \leqslant \mathcal{N}^B$ nor the relation $\mathcal{N}^A \geqslant \mathcal{N}^B$ need hold.

In accord with Theorem 3.3 we make the following definitions:

Partial ordering of Ω. We order the sets $\omega \in \Omega$ by inclusion.

Partial ordering in Θ. We assign two MT-spaces A and B in Θ the order $A \leqslant B$ if (3.8)$'$ holds, or (equivalently) if (3.8)$'''$ holds. We write $A = B$ in Θ if strict equality holds in (3.8)$'$, or (equivalently) in (3.8)$'''$.

Corollary 3.1, taken with Theorem 3.3, gives the following:

COROLLARY 3.2. *The one-to-one mapping $\omega \to A_\omega$ of Ω onto Θ, and the inverse of this mapping preserve partial order of Ω and Θ respectively.*

The above result can be extended by the introduction of equivalence classes, as follows.

The space Ω_1. Let Ω_1 be the ensemble of all non-empty subsets of C-measures ω such that ϱ_ω is finite-valued. We regard two sets ω_1 and

ω_2 in Ω_1 as *equivalent* if $\omega_1^* = \omega_2^*$. Under the mapping $\omega \to A_\omega$ of Theorem 3.1 two equivalent sets ω_1 and ω_2 have the same images $A_{\omega_1} = A_{\omega_2}$ in Θ.

The following two lemmas are needed.

LEMMA 3.1. *If B is an MT-space, and ω a set of C-measures on E such that*

$$\mathcal{N}^B(v) = \varrho_\omega(v) \qquad (v \in \mathcal{K}_C) \tag{3.10}$$

then

$$\mathcal{N}^B(y) = \sup_{\alpha\in\omega} \int |y|\, d\,|\alpha| \qquad (y \in B) \tag{3.11}$$

while $\mathcal{B}^u = \omega^*$ and

$$\mathcal{N}^B(x) = \sup_{\alpha\in\omega^*} \int^* |x|\, d\,|\alpha| \qquad (x \in \boldsymbol{C}^E). \tag{3.12}$$

Proof of (3.12). As in Theorem 3.1 let A_ω be the unique maximal MT-space such that $\mathcal{A}_\omega^u = \omega^*$. In accord with (3.3) and (3.10)

$$\mathcal{N}^{A_\omega}(v) = \mathcal{N}^B(v) \qquad (v \in \mathcal{K}_C)$$

so that it follows from Theorem 3.3 that $\mathcal{B}^u = \mathcal{A}_\omega^u$.

Thus $\mathcal{B}^u = \omega^*$ and (3.12) holds.

Proof of (3.11). Let A be the semi-normed vector subspace of C^E with carrier that of B, and such that

$$\mathcal{N}^A(y) = \sup_{\alpha\in\omega} \int |y|\, d\,|\alpha| \qquad (y \in B).$$

Since

$$\mathcal{N}^B(y) = \sup_{\alpha\in\omega^*} \int |y|\, d\,|\alpha| \qquad (y \in B)$$

we infer that $\mathcal{N}^A(y) \leqslant \mathcal{N}^B(y)$ for $y \in B$. It follows from this relation that not only $\mathcal{N}^B$ but also $\mathcal{N}^A$ is continuous on B in the topology T_B defined by $\mathcal{N}^B$. But $\mathcal{K}_C$ is everywhere dense in B in T_B, and $\mathcal{N}^A|K_C = \mathcal{N}^B|K_C$. It follows that $\mathcal{N}^A|B = \mathcal{N}^B|B$. We infer that (3.11) holds.

LEMMA 3.2. *If ω_1 and ω_2 are two non-empty sets of C-measures such that ϱ_{ω_1} and ϱ_{ω_2} are finite-valued, then the relation $\omega_1^* \subset \omega_2^*$ is equivalent to the relation $\varrho_{\omega_1} \leqslant \varrho_{\omega_2}$, while the relation $\omega_1^* = \omega_2^*$ is equivalent to the equality $\varrho_{\omega_1} = \varrho_{\omega_2}$.*

By virtue of Theorem 3.1 (b)

$$\omega_1^* = \mathcal{A}_{\omega_1}^u, \qquad \omega_2^* = \mathcal{A}_{\omega_2}^u.$$

It follows from Theorem 3.3, applied to A_{ω_1} and A_{ω_2}, that the relation $\omega_1^* \subset \omega_2^*$ is equivalent to the relation $\varrho_{\omega_1^*} \leqslant \varrho_{\omega_2^*}$ (and since $\varrho_{\omega_i} = \varrho_{\omega_i^*}$, $i = 1, 2$) equivalent to the relation $\varrho_{\omega_1} \leqslant \varrho_{\omega_2}$. That the equality $\omega_1^* = \omega_2^*$ is equivalent to the equality $\varrho_{\omega_1} = \varrho_{\omega_2}$ is similarly proved.

Given a non-empty set ω of C-measures on E with ϱ_ω finite-valued, let B be an MT-space such that $\mathcal{B}^u = \omega^*$. Then (3.12) holds in accord with (2.1), and reduces to (3.10) on $\mathcal{K}_C$, thus implying (3.11). It is obvious from these formulas that

$$\sup_{\alpha\in\omega} \int^* |x|\, d\,|\alpha| \leqslant \mathcal{N}^B(x) \qquad (x \in \boldsymbol{C}^E). \tag{3.13}$$

If $\omega = \omega^*$ the equality prevails in (3.13). When $\omega \neq \omega^*$ one naturally raises the question, can the equality prevail in (3.13)? The answer is „yes" for some choices of ω, „no" for others. The following two examples make this clear.

Example 3.1. Let ω consist of a single C-measure β. Then ω^* consists of all C-measures α such that $|\alpha| \leqslant |\beta|$. The MT-space, $B = \mathcal{L}_C^1(\beta)$ is such that $\mathcal{B}^u = \omega^*$ (cf. [7], § 14). The equality prevails in (3.13) with the two members of (3.13) equal to $N_C^1(x, \beta)$.

Example 3.2. Let E_0 be an everywhere dense set in E and let ω be the set of point C-measures $e_t| (t \in E_0)$ such that $|e_t|$ is a measure with unit mass at t. Set

$$\|v\| = \max_{t\in E} |v(t)| = \sup_{t\in E_0} |v(t)| \qquad \text{for} \qquad v \in \mathcal{K}_C.$$

We have

$$\varrho_\omega(v) = \sup_{\alpha\in\omega} \int |v|\, d\,|\alpha| = \sup_{t\in E_0} |e_t|(|v|) = \sup_{t\in E_0} |v(t)| = \|v\|.$$

The semi-norm ϱ_ω is the ordinary semi-norm of $\mathcal{K}_C$. So semi-normed $\mathcal{K}_C$ is an MT-space B such that $\mathcal{B}^u = \omega^*$, in accord with Lemma 3.1. Recall that $\mathcal{B}$ is the set of all C-measures.

We have seen in Theorem 15.3 of [7] that the natural extension $\mathcal{N}^B$ of ϱ_ω is such that

$$\mathcal{N}^B(x) = \sup_{t\in E} |x(t)| \qquad (x \in \boldsymbol{C}^B)$$

while it is clear from the choice of ω that

$$\sup_{\alpha\in\omega} \int^* |x|\, d\,|\alpha| = \sup_{t\in E_0} |x(t)| \qquad (x \in \boldsymbol{C}^E).$$

Thus the two members of (3.13) can be equal for every $x \in \boldsymbol{C}^E$ only if $E = E^0$.

§ 4. The lattice Λ of semi-norms on $\mathcal{K}_C$. Let I be the non-empty range of an index i. Let (A_i), $i \in I$, be a collection of MT-spaces on E. The intersection

$$J = \bigcap_{i\in I} A_i \tag{4.1}$$

(regarded as an intersection of carriers) is a vector subspace of C^E which includes $\mathcal{K}_C$. If J is assigned the trivial semi-norm, $x \to \mathcal{N}(x) = 0$, J is certainly an MT-space. We ask the question, how can J be assigned a non-trivial semi-norm in such a fashion that J is an MT-space? How can this assignment be made to depend upon the collection $[\mathcal{N}^{A_i}]$, $i \in I$, of the semi-norms of the given spaces A_i?

The possibilities here will be illustrated by first considering the special case in which the MT-spaces A_i are the spaces

$$A_i = \mathcal{L}_C^1\ (\beta_i) \qquad (i \in I) \tag{4.2}$$

where β_i is a C-measure on E. Recall that the space of measures on E is an ordered vector space over $\boldsymbol{R}$ which is "completely rediculated" (cf. [5], p. 21 and p. 54). In particular a measure $\mu = \inf_{i \in I} |\beta_i|$ exists. With this understood the following theorem is an analogue of the much more general Theorem 5.1.

Let $\mathcal{N}_C^1(\cdot, \alpha)$ denote the semi-norm with values $N_C^1(x, \alpha)$. Given a measure μ, let μ_e be its extension as a C-measure (see [7], p. 151).

THEOREM 4.1. *Let A_i be an MT-subspace of the space $\mathcal{L}_C^1(\beta_i)$, $i \in I$. Set $J = \bigcap_i A_i$ and*

$$\inf_{i \in I} |\beta_i| = \mu. \tag{4.3}$$

If J is assigned the semi-norm $N_C^1(\cdot, \mu_e)$, then J is an MT-subspace of $\mathcal{L}_C^1(\mu_e)$.

This theorem could be proved as a consequence of Theorem 5.1. A brief proof may be indicated as follows. Each $x \in J$ is μ_e-measurable, since x is β_i-measurable and $|\beta_i| \geqslant \mu$. For such an x

$$\int^* |x|\, d|\mu_e| \leqslant \int |x|\, d|\beta_i|$$

so that x is in $\mathcal{L}_C^1(\mu_e)$. (Theorem 9.4, [7], p. 168). It is clear that J is C-proper, $J \supset \mathcal{K}_C$ and that J admits a semi-norm induced by $N_C^1(\cdot, \mu_e)$. It will follow from Lemma 4.2 that J is an MT-subspace of $\mathcal{L}_C^1(\mu_e)$.

As we have pointed out in § 1 the Bourbaki ordering of measures $\mu \geqslant 0$ implies and is implied by a similar partial ordering of the semi--norms $N_1(\cdot, \mu)$ on K_+. Our purpose will be served by partially ordering the monotone semi-norms on $\mathcal{K}_C$.

The lattice Λ. Let $u \to M(u)$ be an arbitrary monotone seminorm defined for $u \in \mathcal{K}_C$. If M_1 and M_2 are two such semi-norms we write $M_1 \leqslant M_2$ if for each $u \in \mathcal{K}_C$, $M_1(u) \leqslant M_2(u)$, and write $M_1 = M_2$ in case the two mappings M_1 and M_2 are identical. Let Λ denote the partially ordered space of these semi-norms.

For these semi-norms

$$M(u) = M|u| = M(\bar{u}). \tag{4.4}$$

It follows that $M_1 \leqslant M_2$ $(M_1 = M_2)$ if and only if $M_1(f) \leqslant M_2(f)$ $\big(M_1(f) = M_2(f)\big)$ for each $f \in \mathcal{K}_+$.

Bounds in Λ. Let (M_i), $i \in I$, be a set of semi-norms in Λ. The mappings G and g of $\mathcal{K}_C$ into $\bar{R}_+$ with values

$$G(u) = \sup_{i \in I} M_i(u), \qquad g(u) = \inf_{i \in I} M_i(u) \qquad (u \in \mathcal{K}_C) \tag{4.5}$$

will be respectively termed the *superior envelope* and *inferior envelope* of the set (M_i). We write

$$G = \operatorname{env.} \sup_{i \in I} M_i, \qquad g = \operatorname{env.} \inf_{i \in I} M_i. \tag{4.6}$$

Some of the values $G(u)$ may be infinite; all of the values $g(u)$ are finite: It is clear that both G and g are monotone in the sense of § 1, and that for $0 \neq \lambda \in \boldsymbol{C}$,

$$G(\lambda u) = |\lambda| G(u), \qquad g(\lambda u) = |\lambda| g(u). \tag{4.7}$$

Moreover for u and $v \in \mathcal{K}_C$

$$G(u+v) \leqslant G(u) + G(v). \tag{4.8}$$

Simple examples show that g does not in general satisfy a relation similar to (4.8).

Let $\sup_i M_i$ and $\inf_i M_i$ respectively denote the least upper bound and greatest lower bound in Λ of the set (M_i), provided these bounds exist in Λ. This sup and inf in Λ are not to be confused with the env. sup and env. inf in (4.6). The latter need not be in Λ.

LEMMA 4.1 (a). *If the mapping G defined by* (4.5) *has finite values, then G is in Λ and*

$$G = \sup_{i \in I} M_i = \operatorname{env.} \sup_{i \in I} M_i. \tag{4.9}$$

(b) Inf$_i$ M_i *always exists in Λ, but its values are in general inferior to those of g as defined in* (4.5).

Statement (a) is immediate. That $\inf_i M_i$ always exists in Λ follows from (a). For the set of lower bounds of (M_i) in Λ is not empty since it includes the null semi-norm. The env. sup of these lower bounds has finite values, and so by (a), the sup in Λ of these lower bounds exists. By definition of $\inf_i M_i$ in Λ, this sup equals $\inf_i M_i$.

The natural extension of a semi-norm $M \in \Lambda$. Provided with the semi-norm M, $\mathcal{K}_C$ is an MT-space which we denote by M_0. The measure dual $\mathcal{M}_0'$ of M_0 is well-defined, as is the natural extension

$$M(x) = \sup_{\alpha \in \mathcal{M}_0^u} \int^* |x|\, d|\alpha| \qquad (x \in C^E) \tag{4.10}$$

of M over all of C^E. This extension induces a monotone semi-norm over the vector subspace $\mathcal{T}^{M_0}$ of C^E of mappings $x \in C^E$ on which $M(x)$ is finite.

We prepare for § 5 with a lemma.

LEMMA 4.2. *Suppose that H is an MT-space and J a C-proper vector subspace with a monotone semi-norm, and such that as carriers $H \supset J \supset \mathcal{K}_C$. If*

$$\mathcal{N}^J(x) \leqslant \mathcal{N}^H(x) \qquad (x \in J) \tag{4.11}$$

then J is an MT-space.

It is clear that J satisfies Condition I provided $\mathcal{K}_C$ is everywhere dense in J. That $\mathcal{K}_C$ is everywhere dense in J follows form the fact that $\mathcal{K}_C$ is everywhere dense in H, and that the relation (4.11) holds.

We shall now show that J satisfies Conditions II.

If ζ is in J', ζ is continuous on J in the topology induced by $\mathcal{N}^H$ on J, since (4.11) holds. It follows from Corollary 1 of [4], p. 111, that there exists a continuous linear extension η of ζ over H. Hence η is in H' and $\hat{\eta} = \hat{\zeta}$. Since H is an MT-space $\eta(x) = \int x d\hat{\eta}$ for each $x \in H$. In particular for $x \in J$

$$\zeta(x) = \eta(x) = \int x d\hat{\eta} = \int x d\hat{\zeta}.$$

Thus J is an MT-space.

§ 5. Semi-norming $\bigcap_i A_i$ by extension of an inf **in Λ.** We shall generalize Theorem 4.1 in the following way:

THEOREM 5.1. *Let (A_i), $i \in I$, be a non-empty collection of MT-subspaces of C^E with carrier intersection J. The natural extension of the semi-norm*

$$\inf_{i \in I} [\mathcal{N}^{A_i} | \mathcal{K}_C] \qquad (\text{Inf in } \Lambda) \tag{5.1}$$

induces a semi-norm on J, and with this semi-norm J is an MT-space such that

$$\mathcal{G}^u = \bigcap_{i \in I} \mathcal{A}_i^u. \tag{5.2}$$

Let the MT-space with carrier that of $\mathcal{K}_C$ and with semi-norm (5.1) be denoted by Λ. Then, by hypothesis, for fixed i

$$\mathcal{N}^\Lambda(v) \leqslant \mathcal{N}^{A_i}(v) \qquad (v \in \mathcal{K}_C).$$

If $\mathcal{N}^\Lambda$ is naturally extended over C^E it follows from Theorem 3.3, applied to Λ and A_i, that

$$\mathcal{N}^\Lambda(x) \leqslant \mathcal{N}^{A_i}(x) \qquad (x \in J). \tag{5.3}$$

Noting that as carriers $A_i \supset J \supset \mathcal{K}_C$, we shall apply Lemma 4.2. We understand that J has a semi-norm induced by the extended $\mathcal{N}^\Lambda$. Thus J is a C-proper vector subspace of A_i with a monotone semi-norm such that

$$\mathcal{N}^J(x) \leqslant \mathcal{N}^{A_i}(x) \qquad (x \in J) \tag{5.4}$$

in accord with (5.3). Lemma 4.2 implies that J is an MT-space.

Proof of (5.2). Set $\omega = \bigcap_i \mathcal{A}_i^u$. We seek to prove that $\mathcal{G}^u = \omega$. According to Theorem 3.3, applied to J and A_i, the relation (5.4) implies that $\mathcal{G}^u \subset \mathcal{A}_i^u$. Since this is true for each $i \in I$, $\mathcal{G}^u \subset \omega$.

Proof that $\mathcal{G}^u \supset \omega$. Since A_i is an MT-space, formula (2.0) gives

$$\mathcal{N}^{A_i}(v) = \sup_{\alpha \in \mathcal{A}_i^u} \int |v|\, d|\alpha| \qquad (v \in \mathcal{K}_C). \tag{5.5}$$

Now $\omega \subset \mathcal{A}_i^u$, so that we infer from (5.5) that

$$\mathcal{N}^{A_i}(v) \geqslant \sup_{\alpha \in \omega} \int |v|\, d|\alpha| = \varrho_\omega(v) \tag{5.6}$$

by virtue of the definition (3.1) of ϱ_ω. Moreover ϱ_ω is a semi-norm in Λ, and in accordance with (5.6), is a lower bound in Λ of the semi-norms $\mathcal{N}^{A_i}|\mathcal{K}_C$ in Λ. But $\mathcal{N}^J|\mathcal{K}_C$ is by hypothesis the greatest lower bound in Λ of these lower bounds, so that $\varrho_\omega(v) \leqslant \mathcal{N}^J(v)$ for $v \in \mathcal{K}_C$. If one sets $\mathcal{G}^u = \omega_1$, then by (2.0)

$$\mathcal{N}^J(v) = \sup_{\alpha \in \omega_1} \int |v|\, d|\alpha| = \varrho_{\omega_1}(v) \qquad (v \in \mathcal{K}_C).$$

Thus $\varrho_\omega \leqslant \varrho_{\omega_1}$, or equivalently by Lemma 3.2, $\omega^* \subset \omega_1^*$. Since ω_1 is saturated this relation implies that $\omega \subset \omega_1$, that is that $\omega \subset \mathcal{G}^u$ as desired.

Thus $\mathcal{G}^u = \omega$ and the proof of Theorem 5.1 is complete.

In Theorem 4.1 it is clear that the semi-norm assigned to the intersection J is trivial if and only if $\mu = 0$. In the case of general MT-spaces A_i, the following theorem gives an analogous answer as to when the semi-norm defined by (5.1) on $\mathcal{K}_C$, is trivial.

THEOREM 5.2. *A necessary and sufficient condition that the semi-norm* (5.1) *be non-trivial is that the intersection*

$$\bigcap_{i\in I} \mathcal{A}_i^u \tag{5.7}$$

(cf. 5.2) *contain a non-zero* C*-measure.*

Let B be the MT-space with carrier that of $\mathcal{K}_C$ and trivial semi-norm. It follows from Lemma 1.1 that $\mathcal{B}^u$ consists of the null measure alone. Let J be the MT-space of Theorem 5.1. Since $\mathcal{N}^B(v)=0$ for each $v\in\mathcal{K}_C$, the condition that $\mathcal{N}^J$ be non-trivial is equivalent to the condition that the relation $\mathcal{N}^J|K_C \geqslant \mathcal{N}^B|K_C$ not be a strict equality, or by Theorem 3.3 that the inclusion $\mathcal{J}^u \supset \mathcal{B}^{(u)}$ not be an equality. Thus if $\mathcal{N}^J$ is non-trivial $\mathcal{J}^u$ must contain a non-zero C-measure, and conversely.

This establishes the theorem.

§ 6. Semi-norming $\bigcap_i A_i$ by extension of a sup in Λ. Throughout this section we shall be concerned with a non-empty set (A_i), $i\in I$, of MT-subspaces of C^E, and shall set

$$J=\bigcap_i A_i, \qquad V=\bigcup_i A_i \qquad (i\in I). \tag{6.0}$$

If the mapping G of $\mathcal{K}_C$ into $\overline{\boldsymbol{R}}_+$ with values

$$G(v)=\sup_i \mathcal{N}^{A_i}(v) \qquad (v\in\mathcal{K}_C) \tag{6.1}$$

is finite-valued, then in Λ

$$G=\sup_i[\mathcal{N}^{A_i}|\mathcal{K}_C] \qquad (i\in I). \tag{6.2}$$

We set

$$\omega=\bigcup_i \mathcal{A}_i^u \tag{6.3}$$

and state the following lemma:

LEMMA 6.1. *The mapping $v\to G(v)$ defined by* (6.1) *and the mapping $v\to\varrho_\omega(v)$ defined by* (3.1), *where ω is given by* (6.3), *are such that*

$$G(v)=\varrho_\omega(v) \qquad (v\in\mathcal{K}_C). \tag{6.4}$$

There is no assumption in this lemma that G or ϱ_ω is finite-valued.

Taking account of the formula (2.0) applied to A_i, one has

$$\mathcal{N}^{A_i}(v)=\sup_{a\in\mathcal{A}_i^u}\int|v|\,d|a| \qquad (v\in\mathcal{K}_C). \tag{6.5}$$

By virtue of (6.1) and (6.5) respectively,

$$G(v)=\sup_i \mathcal{N}^{A_i}(v)=\sup_i\sup_{a\in\mathcal{A}_i^u}\int|v|\,d|a|=\sup_{a\in\omega}\int|v|\,d|a|$$

since ω is given by (6.3). This establishes (6.4).

The possibilities of semi-norming J by the natural extension of the semi-norm G, when the sup (6.2) exists, are greatly clarified by showing that the conjecture which a priori seems most natural, is indeed false. The conjecture which we shall disprove is as follows:

(Z) *If G, as defined by* (6.2), *exists in Λ, and can be naturally extended with finite values over J, then J, semi-normed with this extension, is an* MT*-space.*

That (Z) is false is shown by the following example.

Example 6.1. We define a set (A_n), $n=1,2,\ldots$, of MT-spaces, with intersection J and such that the

$$\sup_n[\mathcal{N}^{A_n}|\mathcal{K}_C] \tag{6.6}$$

exists in Λ and admits a finite-valued natural extension on J. Semi-normed by this extension J will be shown to fail to satisfy Condition I.

E and E_0. Let E be the interval [0,1] with euclidean topology. Let E_0 be an everywhere dense set (t_n) of points $t_n\in E$, $n=1,2,\ldots$ Let ε_n denote the measure of mass 1 at the point t_n. If $f\in R_+^E$ then (see [5], p. 109)

$$\varepsilon_n^*(f)=\int^* f\,d\varepsilon_n=f(t_n).$$

Let e_n be the C-measure extending ε_n so that $|e_n|=\varepsilon_n$.

The carrier J. To construct Example 6.1 let J be the set of bounded mappings in C^E.

The spaces A_n. Each space A_n, $n=1,2,\ldots$, shall be a vector subspace of C^E with J as carrier, so that $J=\bigcap_n A_n$. We assign A_n a semi-norm

$$x\to|x(t_n)| \qquad (x\in J). \tag{6.7}$$

We must show that A_n is an MT-space.

It is clear that A_n is C-proper, that its semi-norm is monotone, that $A_n\supset\mathcal{K}_C$ and that $\mathcal{K}_C$ is everywhere dense in A_n. Thus A_n satisfies Condition I. To show that A_n is an MT-space we have merely to apply Corollary 2.2 to A_n. According to Lemma 1.1, $\mathcal{A}_n^u$ consists of the C-measures a such that $|a(u)|\leqslant|u(t_n)|$. Hence $|e_n|\geqslant|a|$ for each $a\in\mathcal{A}_n^u$, so that

$$\sup_{a\in\mathcal{A}_n^u}\int^*|x|\,d|a|=\int^*|x|\,d|e_n|=|x(t_n)| \tag{6.8}$$

for $x \in C^E$. Thus the condition (2.6) of Corollary 2.2, applied to A_n, takes the form,

$$\mathfrak{N}^{A_n}(x) = |x(t_n)| \quad (x \in J).$$

It is satisfied by virtue of (6.7). Hence A_n is an MT-space.

Properties of G. For $v \in \mathcal{K}_C$ let $v \to \|v\|$ be the classical semi-norm. Applying (6.1) to the ensemble (A_n) we have

$$G(v) = \sup_n N^{A_n}(v) = \sup_n |v(t_n)| = \|v\|,$$

since the set (t_n) is everywhere dense in [0,1]. (Z) calls for the natural extension of G over J. As we have shown in Theorem 15.3 of [7], the natural extension of G over C^E has the values

$$G(x) = \sup_{t \in E} |x(t)| \quad (x \in C^E). \tag{6.9}$$

This extension has finite values for x in J. Thus all the conditions of (Z) are satisfied by the ensemble (A_n) of MT-spaces and their intersection J, as semi-normed by the extended G. But so semi-normed J is not an MT-space, since $\mathcal{K}_C$ is not everywhere dense in J. For the closure of $\mathcal{K}_C$ in the uniform topology defined by (6.9) includes no discontinuous mappings.

Thus (Z) as stated is false.

The following modification of (Z) is true. In it we refer to the spaces A_i, $J = \bigcap_i A_i$ and set $\omega = \bigcup_i A_i^u$, as previously defined.

THEOREM 6.1. *If the semi-norms*

$$[\mathfrak{N}^{A_i}|J], \quad i \in I, \tag{6.11}$$

are majorized by the semi-norm $\mathfrak{N}^H$ of an MT-space H with carrier J, then G, as defined in (6.2), exists as a semi-norm in A, and admits a finite natural extension over J. Semi-normed with this extension, J is an MT-space and $\mathcal{G}^u = \omega^*$.

Proof that J is an MT-space. It is clear that $G(v) \leqslant \mathfrak{N}^H(v)$ $(v \in \mathcal{K}_C)$.

It follows from Theorem 3.3 that the natural extension of G is at most the natural extension of $\mathfrak{N}^H$ on C^E, so that $\mathfrak{N}^J \leqslant \mathfrak{N}^H$ on J. That J is an MT-space follows from Lemma 4.2, applied to H and J, noting that as carriers $H = J \supset \mathcal{K}_C$, that J is C-proper, and that J has a monotone semi-norm such that $\mathfrak{N}^J \leqslant \mathfrak{N}^H$ on J.

Proof that $\mathcal{G}^u = \omega^*$. By the definition of $\mathfrak{N}^J$, as given in the theorem,

$$\mathfrak{N}^J(v) = G(v) \quad (v \in \mathcal{K}_C). \tag{6.10}$$

It follows from (6.4) that $N^J(v) = \varrho_\omega(v)$, and then from Lemma 3.1, applied to J, that $\mathcal{G}^u = \omega^*$.

This completes the proof of the theorem.

MT-spaces J of Cauchy type. Spaces J of this type are defined in § 6 of [9]. One of their fundamental properties is that they have the form $J = \bigcap_i A_i$, $i \in I$. More explicitly an MT-space J of Cauchy type has the form (see [9], § 7)

$$J = \bigcap_{a \in \mathcal{G}'} \mathcal{L}_C^1(a). \tag{6.12}$$

This space J obviously has the representation

$$J = \bigcap_{a \in \mathcal{G}^u} A_a \quad \big(A_a = \mathcal{L}_C^1(a)\big). \tag{6.13}$$

In accordance with (2.0), for $v \in \mathcal{K}_C$

$$\mathfrak{N}^J(v) = \sup_{a \in \mathcal{G}^u} \int |v|\, d|a| = \sup_{a \in \mathcal{G}^u} N_C^1(v, a).$$

Thus

$$\mathfrak{N}^J|\mathcal{K}_C = \sup_a [\mathfrak{N}^{A_a}|\mathcal{K}_C] \quad (a \in \mathcal{G}^u) \tag{6.14}$$

where the sup in (6.14) is taken in A, with $\mathcal{G}^u$ affording the range I of a.

The conditions of Theorem 6.1 are satisfied by the above set (A_a), $a \in \mathcal{G}^u$, of MT-spaces. The conclusions of Theorem 6.1 in this case present nothing essentially new concerning J, but a review of these conditions and conclusions in this case show the workings of the theorem. The semi-norms (6.11) are here majorized by the semi-norm $\mathfrak{N}^J$ of the given MT-space. The semi-norm G, as defined in (6.2), is here given by (6.14), and admits a finite natural extension, namely $\mathfrak{N}^J$, over J. For fixed $a \in \mathcal{G}^u$ the set of C-measures $\mathcal{A}_a^u$ is the set of all C-measures β such that $N_C^1(\cdot, \beta) \leqslant N_C^1(\cdot, a)$ (cf. Lemma 1.1). By the definition (6.3)

$$\omega = \bigcup_a \mathcal{A}_a^u \quad (a \in \mathcal{G}^u).$$

The conclusion of Theorem 6.1 that $\mathcal{G}^u = \omega^*$ is independently verifiable.

Every MT-space J of the form (6.12) is maximal.

To verify this note that J and the closure $\bar{J}$ of J in $\mathcal{F}^J$ have the same measure dual $\mathcal{G}'$, since they induce the same MT-subspace with carrier $\mathcal{K}_C$. If follows that

$$\bar{J} \subset \bigcap_{a \in \mathcal{G}'} \mathcal{L}^1_C(a) \tag{6.15}$$

since each $x \in \bar{J}$ is a-integrable for each $a \in \mathcal{G}'$. From (6.12) and (6.15) we infer that $\bar{J} \subset J$, and hence that $\bar{J} = J$.

The question is open as to whether every MT-space J of the form (6.12) is of Cauchy type.

We state a theorem which concerns

$$V = \bigcup_i A_i, \quad \omega = \bigcup_i \mathcal{A}_i^u,$$

A_ω as defined in Theorem 3.1, and G as defined in (6.2).

THEOREM 6.2. *If ϱ_ω is finite-valued the semi-norm G exists in A, and has a finite natural extension over $H = V \cap A_\omega$ by virtue of which H is an MT-space with $\mathcal{N}^u = \omega^*$.*

Since $H = V \cap A_\omega \subset A_\omega$, a semi-norm can be induced on H by $\mathcal{N}^{A_\omega}$, the natural extension of ϱ_ω. Or equivalently (Lemma 6.1) $\mathcal{N}^H$ can be induced by the natural extension of G. That H so semi-normed, is an MT-space follows from Lemma 4.2, applied to A_ω and H, on noting that as carriers $A_\omega \supset H \supset \mathcal{K}_C$, that H is C-proper, and has a semi-norm induced by $\mathcal{N}^{A_\omega}$. Since

$$\mathcal{N}^H(v) = \mathcal{N}^{A_\omega}(v) \qquad (v \in \mathcal{K}_C)$$

$\mathcal{N}^u = \mathcal{A}^u_\omega$ by Theorem 3.3. By Theorem 3.1 $\mathcal{A}^u_\omega = \omega^*$. Hence $\mathcal{N}^u = \omega^*$, completing the proof of the theorem.

The above theorem and proof remain valid if V is replaced by $J = \bigcap_i A_i$.

There are two special cases worthy of note. If the range I of i is finite the values of G given by (6.1) are finite, so that G exists as a sup in A and ϱ_ω is finite-valued by (6.4). In this case Theorem 6.2 can be restated, omitting the hypothesis that ϱ_ω is finite-valued.

The second special case arises when $\bigcap_i A_i = \mathcal{K}_C$. In this case (Z) is true. For G exists in A by hypothesis of (Z), and if G is assigned to $\mathcal{K}_C$ as a semi-norm an MT-space results, as affirmed.

REFERENCES

[1] N. Bourbaki, *Eléments de Mathématique, Théorie des ensembles*, Fasc. de Résultats, Paris 1951.

[2] — *Eléments de Mathématique, Topologie Générale*, Chap. I-II, Paris 1951.

[3] — *Eléments de Mathématique, Topologie Générale*, Chap. IX, Paris 1948.

[4] — *Eléments de Mathématique, Espaces Vectoriels Topologiques*, Chap. I-II, Paris 1953.

[5] — *Eléments de Mathématique, Intégration*, Chap. I-IV, Paris 1952.

[6] — *Eléments de Mathématique, Intégration*, Chap. V, Paris 1956.

[7] — M. Morse and W. Transue, *Semi-normed vector space with duals of integral type*, Journal d'Analyse Mathématique 4 (1955), p. 149-186.

[8] — *Products of a C-measure and a locally integrable mapping*, Canadian Journal of Mathematics 9 (1957), p. 475-486.

[9] — *Vector subspaces A of C^E with duals of integral type*, Journal de Mathématiques Pures et Appliquées (9) 37 (1958).

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