some C_y or $\mathfrak{U}_{x<Oy}\, C_y$? More generally, how do the entire hierarchies compare for different initial classes with specified mode of generation (or in other terms, for different ways of generating a class from an assumed function, which is $\lambda ba\, 0$ for the lowest class)? In particular, how much smaller a class than the primitive recursive functions can one start with and get the same $\mathfrak{U}_{y\varepsilon O}\, C_y$?

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LOCAL ORIENTABILITY

BY

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It is the purpose of this paper* 1° to clarify and extend the notion of *local orientability* which was defined on p. 281-282 of my book [7], and 2° to apply the results obtained to establish a definition of orientability for an n-dimensional generalized manifold ($= n$-gm) which is the exact analogue of the Poincaré definition [1].

It is hoped that these results will contribute to the solutions of a number of unsolved problems concerning manifolds (see, for example, [7], p. 382-383, problems 4.1, 4.5).

1. Some basic lemmas. For the proofs given below it is necessary to have the following definitions and lemmas, which are inserted at this point for convenience of reference.

1.1. LEMMA. *In an n-dimensional space S, if P is an open set with compact closure, and γ^n is a cycle mod $S-P$, then there exists a minimal closed (rel. P) subset F of P such that γ^n is carried by $F \cup (S-P)$.*

Proof. The portion of γ^n on $\bar{P}$ is a cycle Z^n mod $F(P)$ on $\bar{P}$. As $\bar{P}$ is compact, there exists by [7], p. 205-6, Lemma 2.3, a minimal closed subset F' of $\bar{P}$ that contains $F(P)$ such that $Z^n \sim 0 \bmod F'$; and by [7], p. 206, Lemma 2.6, F' is unique and a closed carrier of Z^n. Let $F = F' \cap P$. Since $\gamma^n \sim Z^n \bmod S-P$, the lemma follows.

1.2. LEMMA. *If S is an n-dimensional locally compact space, then every infinite cycle Γ^n of S has a unique minimal closed carrier.*

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Terminology and notation are that of my book [7].

[1] This definition states that an n-manifold, without boundary, whose elements are oriented k-cells ($k = 0, 1, \ldots, n$) is orientable if every "closed chain" of cells $\sigma_i^n, \sigma_{j(1)}^{n-1}, \sigma_{k(1)}^n, \ldots, \sigma_{k(m)}^n, \sigma_{j(m+1)}^{n-1}, \sigma_{k(m+1)}^n, \ldots, \pm\sigma_i^n$ in which $\sigma_{k(m)}^n$ and $\sigma_{k(m+1)}^n$ are oppositely related to $\sigma_{j(m+1)}^{n-1}$ had $+\sigma_i^n$ as „end" element. See [6], § 8.

Proof. If Γ^n is on a compact subset of S, then the lemma follows from [7], p. 206, Lemma 2.6. If Γ^n is not on a compact set, let us compactify S by the addition of a single ideal point p^*; let $S^* = S \cup p^*$. Then there is a cycle mod p^* of S^*, which we call Γ^{n*}, such that $\Gamma^{n*} \sim \Gamma^n$ mod p^* (2).

By Lemma 1.1, there exists a minimal closed (rel. S) subset F of S such that Γ^{n*} is carried by $F \cup p^*$. Since S is n-dimensional, we may assume that $\Gamma^{n*} \sim \Gamma^n$ mod p^* implies $\Gamma^{n*} = \Gamma^n$ mod p^*. It follows that F is a minimal closed carrier of Γ^n.

1.3. Definition. If x is a point of a space S such that $p_n(x) = 1$ (3) then a *canonical pair P, Q of neighborhoods of x (rel. n and the local Betti number)* (cf. [7], p. 192, 6.11) together with a compact cocycle Z_n in Q such that $Z_n \nsim 0$ in P will be called a *canonical triad* of x and will be denoted by the symbol $(x; P, Q, Z_n)$.

1.4. Lemma. *If a locally compact space S is lc at $x \in S$, and $p_n(S, x) = 1$, then there exists a canonical triad $(x; P, Q, Z_n)$ such that P and Q are arbitrarily small connected sets and $P \supset \bar{Q}$.*

2. The n-gm; orientable and locally orientable. By an n-gm ($=$ n-dimensional generalized manifold) we shall mean a space S satisfying the following axioms:

(1) S is a locally compact Hausdorff space.

(2) S is of dimension n (in the Lebesgue sense).

(3) For every $x \in S$, $p_i(x) = 0$ for $i = 1, \ldots, n-1$.

(4) For every $x \in S$, $p_n(x) = 1$.

(5) If F is a proper closed subset of S, then every infinite n-cycle on F bounds on S; or, which is equivalent in the presence of (1), $\mathfrak{H}^n(F) = 0$.

(6) S is connected.

We shall denote the system of axioms (1)-(6) by $\sum_n$ (4).

If an n-gm is compact, we call it an n-gcm ($=$ n-dimensional generalized closed manifold). If it carries a non-bounding n-cycle (compact or infinite), we call it *orientable*.

Definition. An n-gm S is called *locally orientable* if $x \in S$ implies the existence of an open set $U(x)$ such that $U(x)$ is an orientable n-gm, (compare [7], p. 281, 6.1).

(2) As was suggested in [7], p. 246, footnote 1), such cycles as Γ^{n*} could be used to replace the infinite cycles used in [7]. H. Cartan studied groups based on such cycles in [2].

(3) By $p^n(x)$, $p_n(x)$ we denote local Betti and local co-Betti numbers at x (see [7], p. 190, 6.6). If it is desired to designate the space involved, we use the symbols $p^n(S, x)$, $p_n(S, x)$.

(4) It will be observed that $\sum_n$ incorporates Axioms A, B and C on page 244 of [7] and Axiom D′ on page 254 of [7].

Remark. Obviously every orientable n-gm is locally orientable.

2.1. Lemma. *Let P be a non-empty open subset of an n-gm S. Then the injection mapping j_P of $\mathfrak{H}^n(S)$ in $\mathfrak{H}^n(P)$ is an isomorphism into; and consequently if S is orientable, $\mathfrak{H}^n(P) \neq 0$.*

Proof. The sequence

$$\to \mathfrak{H}^n(S-P) \overset{i_s}{\to} \mathfrak{H}^n(S) \overset{j_P}{\to} \mathfrak{H}^n(P) \to$$

is exact. By Axiom (5) of $\sum_n$, $i_s\,\mathfrak{H}^n(S-P) = 0$.

2.2. Theorem. *If P is a connected open subset of an orientable n-gm S, then P is an orientable n-gm.*

Proof. By Lemma 2.1, if P is an n-gm then it is an orientable n-gm. Evidently P satisfies Axioms (1)-(4) and (6) of $\sum_n$, so that we need only prove Axiom (5) for P. Let F be a closed (rel. P) proper subset of P and suppose $\mathfrak{H}^n(F) \neq 0$. By Lemma 1.2 we may assume F to be the closed minimal carrier of a cycle Γ^n of some non-zero element of $\mathfrak{H}^n(F)$. Let Z^n be a fundamental cycle of S and let $p \in F \cap \overline{(P-F)}$. By Axiom (4) of $\sum_n$ there exist neighborhoods V and W of p such that $P \supset V \supset W$ and $p^n(p; V, W) = 1$. Hence there exists a relation $a\Gamma^n \sim bZ^n \bmod S-W$. But $a \neq 0$, since the contrary would imply Z^n homologous to a cycle on a proper closed subset $(S-W)$ of S and thus ~ 0 on S by Axiom (5). But then $b = 0$, else a similar conclusion follows (for $W-F \neq 0$). Hence $\Gamma^n \sim 0 \bmod S-W$. But on a cofinal family of n-dimensional coverings this implies $\Gamma^n = 0$ in W and F not a minimal carrier of Γ^n.

2.2a. Corollary. *Each point of a locally orientable n-gm S is contained in arbitrarily small open sets that are orientable n-gms.*

(Recall that by [7], p. 244, 1.1, an n-gm is locally connected.)

2.3. Theorem. *A necessary and sufficient condition that an n-gm S be locally orientable is that if $x \in S$ then there exist arbitrarily small open sets P containing x such that if F is a closed (rel. P) proper subset of P, then $\mathfrak{H}^n(F) = 0$.*

Proof. The necessity follows from Corollary 2.2a and Axiom (5) of $\sum_n$. For the sufficiency suppose $x \in S$ and U an open set containing x such that $p^n(x; P) = 1$ for all open sets P such that $x \in P \subset U$ (5); in particular, for any such P, $\mathfrak{H}^n(P) \neq 0$. Let P be such an open set, having also the properties of the set P of the statement of the theorem, and suppose $P = P_1 \cup P_2$ separate. But this is impossible, since $\mathfrak{H}^n(P)$ would be the direct sum of $\mathfrak{H}^n(P_1)$ and $\mathfrak{H}^n(P_2)$ each of which is zero by hypo-

(5) See p. 191 of [7]; in both lines 5 and 7 of the page referred to, the word "all" should be followed by the words "arbitrarily small".

thesis. It follows that P is connected and satisfies all axioms of $\sum_n$; and as $\mathfrak{H}^n(P) \neq 0$, P is an orientable n-gm.

2.4. THEOREM. *Let S be an n-gm. Then a necessary and sufficient condition that S be locally orientable is that every connected open subset of S be an n-gm.*

Proof. The condition is sufficient. Let $x \varepsilon S$, and $(x; P, Q, Z_n)$ a canonical triad (Lemma 1.4). By hypothesis, P is an n-gm. And since $Z_n \not\sim 0$ in P, $\mathfrak{H}^n(P) \neq 0$, so that P is an orientable n-gm.

The condition is necessary. Let P be a nonempty, connected open subset of the locally orientable n-gm S, and suppose F is a closed (rel. P) proper subset of P such that $\mathfrak{H}^n(F) \neq 0$. By Lemma 1.2 we may assume F to be a minimal closed carrier of a cycle Γ^n of some non-zero element of $\mathfrak{H}^n(F)$. Since P is connected, there exists an $x \varepsilon F \cap \overline{(P-F)}$. By Theorem 2.3, there exists in P a neighborhood U of x such that if M is a closed (rel. U) proper subset of U, then $\mathfrak{H}^n(M) = 0$. But $F \cap U$ is such an M and it follows that F is not minimal.

2.4a. COROLLARY. *The components of any open subset of a locally orientable n-gm are all n-gms.*

Δ_n An axiom system for locally orientable n-gms.

Let us now consider the following system of axioms:

(1)-(4). These are as in $\sum_n$.

(5). If $x \varepsilon S$ and U an open set containing x, then there exists an open set P such that $x \varepsilon P \subset U$ and such that if γ^n is a cycle mod $S-P$ on a set F such that $F \cap P$ is a closed (rel. P) proper subset of P, then $\gamma^n \sim 0 \bmod S-P$; or, which is equivalent in the presence of (1), $\mathfrak{H}^n(F) = 0$.

(6). As in $\sum_n$.

2.5. THEOREM. *A necessary and sufficient condition that a space S be a locally orientable n-gm is that it satisfy the axiom system Δn.*

Proof. The necessity follows from Theorem 2.3. For the sufficiency it is only necessary to show that S is an n-gm, since then axiom (5) of Δ_n will imply, by virtue of Theorem 2.3, that S is locally orientable. To show that S is an n-gm, it is only necessary to prove that axiom (5) of $\sum_n$ is satisfied. Suppose, then, that there exists a proper closed subset F of S carrying a nonbounding cycle Γ^n. By virtue of Lemma 1.2, we may suppose F to be a minimal closed carrier of Γ^n. Since S is connected, there exists $x \varepsilon F \cap \overline{S-F}$. Let P be a set satisfying axiom (5) of Δ_n. Then $\Gamma^n \sim 0 \bmod S-P$. However, as S is n-dimensional, we may conclude that this implies $\Gamma^n = 0 \bmod S-P$. But this contradicts the fact the F is a minimal closed carrier of Γ^n.

2.6. Axiom (5) of Δ_n is clearly an "in the small" prototype of axiom (5) of $\sum_n$. The axioms of Δ_n form a convenient set with which to define locally orientable n-gms, since they already imply, according to Theorem 2.5, the "in the large" form of the axiom — that is, axiom (5) of $\sum_n$.

2.7. Remark. The question raised in [7], p. 382, Problem 4.1, as to whether every n-gm is locally orientable, evidently reduces, as a result of the above, to the question whether $\sum_n$ implies the „in the small" axiom (5) of Δ_n (P **240**).

3. Alternatives to axiom (5) **of** Δ_n. Although axiom (5) of Δ_n is perhaps ideal, from an axiomatic viewpoint, for incorporating the local orientability property in the definition of n-gm, there are other properties that are often more useful in applications. Consider, for example, the following properties:

3.1. If $x \varepsilon S$, then there exists an open set P containing x and a cycle $\Gamma^n \bmod S-P$ such that $\Gamma^n \not\sim 0 \bmod S-U$ for every non-empty open subset U of P.

3.1s. If $x \varepsilon S$, then there exist arbitrarily small open sets P containing x and cycles $\Gamma^n(P) \bmod S-P$ such that $\Gamma^n(P) \not\sim 0 \bmod S-U$ for every nonempty open subset U of P.

3.2. If $x \varepsilon S$, then there exists an open connected set P containing x and a compact cocycle Z_n in P such that $Z_n \not\sim 0$ in P, and such that if U is any nonempty open subset of P there exists a compact cocycle γ_n in U such that $\gamma_n \sim Z_n$ in P.

3.2s. This is the same as 3.2 with the words "arbitrarily small" inserted between "an" and "open".

3.2s′. If $x \varepsilon S$, there exist arbitrarily small open sets P and Q such that $x \varepsilon Q \subset P$ and such that if U is a non-empty open subset of Q, there exists in U a cocycle $Z_n \not\sim 0$ in P.

3.3. If A is a closed subset of S and $x \varepsilon A \cap \overline{S-A}$, then $p_n(A, x) = 0$.

3.4. If Z^{n-1} is a cycle on a compact subset M of S and F a compact set minimal relative to carrying the homology $Z^{n-1} \sim 0$ on $F \supset M$, then $F-M$ is open in S.

3.4s. If $x \varepsilon S$, then there exists an arbitrarily small open set P containing x such that if Z^{n-1} is a cycle on a compact subset M of P and F is a compact set minimal relative to carrying the homology $Z^{n-1} \sim 0$ on $F \supset M$, then $F-M$ is open in S.

Properties 3.1s, 3.2s, 3.4s are of course localizations of 3.1, 3.2, 3.4 respectively. Property 3.1 was given in [7], p. 281, D″; the word "non-empty" was inadvertently omitted in the statement D″. Property 3.3 was used by Čech [3], p. 686, in defining an n-gm; attention was called to it in [7], p. 289 (bibliographical comment concerning § 6), with the remark that "it seems probable that the local orientability property of an n-gm is equivalent to" property 3.3. Also, property 3.4s is analogous to a property used earlier by Čech [4], p. 644, D_1.

3.5. THEOREM. *If S is a space satisfying axioms* (1)-(4), (6) *of* $\sum_n$, *then axiom* (5) *of* Δ_n *is equivalent to each of the properties* 3.1-3.4s.

Proof. Let S be a space satisfying axioms (1)-(4), (6) of $\sum_n$. Then axiom (5) implies that S has property 3.1s (proof is left to reader); and that 3.1s implies 3.1 is trivial.

To show that 3.1 implies 3.2, let P be an open set satisfying 3.1; since S is lc, we may take for the P of 3.2 the component of the former "P" — the conclusion of 3.1 still holds. We assert P satisfies the axioms of $\sum_n$. To show this, we have only to prove that P satisfies axiom (5) of $\sum_n$. Let F be a proper closed (rel. P) subset of P and γ^n an infinite cycle on F; we may assume F minimal by Lemma 1.2. Let $x \varepsilon F \cap \overline{P-F}$, and U, V open sets such that $x \varepsilon V \subset U \subset P$ and $p^n(x; U, V) = 1$. There exists a homology $a\gamma^n \sim b\Gamma^n \bmod S-V$. Neither a nor b can be zero, since neither γ^n nor Γ^n bounds $\bmod S-V$. But this is impossible, since it implies $\Gamma^n \sim a/b\ \gamma^n \sim 0 \bmod S-(V-F)$, whereas $\Gamma^n \nsim 0 \bmod S-(V-F)$. Thus P is an orientable n-gm, and the conclusion of 3.2 follows from the properties of orientable n-gms (see [7], p. 255, 5.3).

To show that 3.2 implies 3.2s, we need only notice that in the presence of axioms (1)-(4) and (6) of $\sum_n$, the set P of 3.2 is an orientable n-gm (an argument like that of the preceding paragraph shows this, for example), and 3.2s then follows by application of corollary 2.2a above. And that 3.2s implies 3.2s′ is trivial.

Let S satisfy 3.2s′, together with axioms (1)-(4) and (6) of $\sum_n$, and let A, x be as in 3.3. By 3.2s′ there exist arbitrarily small open sets P, Q containing x satisfying the conclusion of 3.2s′. But P and Q may be selected so that in addition $p_n(x; P, Q) = 1$, and there will then (by 3.2s′) exist a cocycle Z_n in $Q-A$ such that $Z_n \nsim 0$ in P. Any cycle $\gamma^n \bmod S-P$ on A must then bound $\bmod S-Q$ on A since Z_n may be taken as a base for cocycles of Q relative to cobounding in P. It follows that $p_n(A, x) = 0$.

With S satisfying 3.3, together with axioms (1)-(4) and (6) of $\sum_n$, let Z^{n-1}, M, F be as in 3.4. Suppose there exists a point x of $F-M$ which is a limit point of $S-(F-M)$. Then $x \varepsilon F \cap \overline{S-F}$, and by 3.3, $p_n(F, x) = 0$. But if P, Q are open sets such that $x \varepsilon Q \subset P \subset S-M$ and such that $p_n(x; F \cap P, F \cap Q) = 0$, it follows from the "working lemmas" ([7], p. 201, 1.4, [7], p. 202, 1.9, and [7], p. 201, 1.3), in this order, that $Z^{n-1} \sim 0$ on $F-Q$. This contradicts the minimal character of F.

That 3.4 implies 3.4s is trivial.

Finally, suppose S satisfies 3.4s, together with axioms (1)-(4) and (6) of $\sum_n$. If U is any open set containing x let P be an open set as in 3.4s and such that $P \subset U$; we may suppose $\overline{P}$ compact. Let Q be a connected open set such that $x \varepsilon Q$, $\overline{Q} \subset P$, and n-cycles on $\overline{Q}$ bound on $\overline{P}$ ([7], p. 196, 7.9) and suppose γ^n is a cycle $\bmod S-Q$ on a set F such that $F \cap Q$ is a closed (rel. Q) proper subset of Q. By Lemma 1.1 (taking the portion of γ^n on Q), we may suppose that $F' = F \cap Q$ is a minimal closed (rel. Q) subset of Q such that γ^n is carried by $M = F' \cup F(Q)$. Suppose $F' \neq 0$.

Now $\partial\gamma^n$ is a cycle on $F(Q)$ such that $\partial\gamma^n \sim 0$ on M ([7], p. 200, 1.1). Also, M is a compact set minimal relative to carrying the homology $\partial\gamma^n \sim 0$ on $M \supset F(Q)$. For suppose there exists a closed set $K \supset F(Q)$ such that $\partial\gamma^n \sim 0$ on K and K is a proper subset of M. Then there exists $\Gamma^n \bmod F(Q)$ on K such that $\partial\Gamma^n \sim \partial\gamma^n$ on $F(Q)$ ([7], p. 201, 1.4); and hence an absolute cycle Z^n on $\overline{Q}$ such that $Z^n \sim \gamma^n - \Gamma^n \bmod S-Q$ ([7], p. 201, 1.6). By the choice of Q, $Z^n \sim 0$ on S. We may restrict ourselves to n-dimensional coverings on the compact subsets of S, and therefore $Z^n \sim 0$ implies $Z^n = 0$, which in turn implies $\Gamma^n = \gamma^n \bmod S-Q$. But this contradicts the fact that K is a proper closed subset of M, so that we conclude M must be minimal relative to carrying the homology $\partial\gamma^n \sim 0$ on $M \supset F(Q)$. Then by 3.4s, $M-F(Q)$ is open in S, implying that $F' = Q$. But this contradicts the choice of F, and we conclude $F' = 0$; *i. e.*, $\gamma^n \sim 0 \bmod S-Q$.

4. A chain condition for local orientability. In [7], p. 251, ff, there is given a condition for orientability of an n-gcm, due to Begle [1], in terms of elements of canonical triads. An analogous condition for local orientability may be given in the following manner:

If S is an n-gm and P and Q a canonical pair of neighborhoods of S (rel. n and the local Betti number), such that $\overline{P}$ is compact and $\overline{Q} \subset P$, then for any covering $\mathfrak{C}$ of S by open sets such that $\mathrm{St}(\overline{Q}, \mathfrak{C}) \subset P$, there exist refinements $\mathfrak{C}_1$ and $\mathfrak{C}_2$ of $\mathfrak{C}$ such that $\mathfrak{C}_1 \cap P$ is finite and each element of $\mathfrak{C}_2$ that meets P is a "Q" of a canonical pair of neighborhoods "P, Q" (relative n and the local Betti number), whose corresponding P is an element of $\mathfrak{C}_1$; and such that each element of $\mathfrak{C}_1$ that meets P is a "P" corresponding to one of the „Q's" of $\mathfrak{C}_2$. (When convenient to do so, we may assume that if $E_1 \varepsilon \mathfrak{C}_1$, $E_2 \varepsilon \mathfrak{C}_2$ so correspond, then $\overline{E}_2 \subset E_1$). To each such canonical pair $E_1, E_2, E_1 \varepsilon \mathfrak{C}_1, E_2 \varepsilon \mathfrak{C}_2$, such that $E_2 \cap \overline{Q} \neq 0$, corresponds a canonical triad $(x; E_1, E_2, Z_n)$. If $(x_i; E_{i1}, E_{i2}, Z_n^i)$ and $(x_j; E_{j1}, E_{j2}, Z_n^j)$ are two such triads such that $E_{i2} \cap E_{j2} \neq 0$, there exists a canonical triad $(y; U, V, Z_n)$ such that $U \subset E_{i2} \cap E_{j2}$, and relations $a_i Z_n^i \sim Z_n$ in E_{i1}, $a_j Z_n^j \sim Z_n$ in E_{j1}, implying $a_i Z_n^i \sim a_j Z_n^j$ in $E_{i1} \cup E_{j1}$.

4.1. THEOREM. *If S is a space satisfying axioms* (1)-(4), (6) *of* $\sum_n$, *then the local orientability axiom* (5) *of* 2.4 *is equivalent to the assertion that: If $x\varepsilon S$, then there exist arbitrarily small connected open sets P, Q such that $x\varepsilon Q \Subset P$ and such that for every covering $\mathfrak{C}$ of S by open sets there exist coverings $\mathfrak{C}_1$, $\mathfrak{C}_2$ and cocycles Z_n^i as defined above, such that for any choice of canonical pairs U, V in the intersections of elements of $\mathfrak{C}_2$ that meet $\bar{Q}$, the ratios a_i/a_j are all* 1.

Proof. By Theorem 2.5, if S satisfies axiom (5) then it is locally orientable, and the existence of the sets P, Q, etc. follows from the existence of an orientable n-gm M containing x and fundamental cocycles of M (compare the first paragraph of the proof of Theorem VIII 3.5 of [7], p 251).

Conversely, suppose that for each $x\varepsilon S$ the sets P, Q, etc., of the "assertion" of the theorem exist. Then to show S satisfies (5) of 2.4 it is only necessary, by virtue of Theorem 3.5, to show that S has one of the properties 3.1-3.4s. Let us show that S has property 3.4s. Let $x\varepsilon S$, and P and Q as in the "assertion" of the theorem. Let P' be an open set such that $x\varepsilon P' \subset Q \subset P$ and such that every compact $(n-1)$-cycle of P' bounds in Q (see [7], p. 196, 7.9). Let Z^{n-1} be a cycle on a compact subset M of P' and let $F \subset Q$ (6) be a closed set minimal with respect to carrying the homology $Z^{n-1} \sim 0$ on F and $F \supset M$ ([7], p. 206, 2.8). Suppose $F-M$ not open in S and let $y\varepsilon F-M$ be a limit point of $S-F$.

Let R_1 be a connected open subset of $Q-M$ containing y, R_2 an open set such that $y\varepsilon R_2 \Subset R_1$, and $p\varepsilon R_2-F$. Let $\mathfrak{C}$ be a covering of S such that $\mathrm{St}(\bar{R}_2, \mathfrak{C}) \subset R_1$, no element of $\mathfrak{C}$ that contains p meets F, and $\mathrm{St}(\bar{Q}, \mathfrak{C}) \subset P$. Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be as in the "assertion".

There exists a simple chain of elements of $\mathfrak{C}$ from y to p in R_1 ([7], p. 33), of which let E_{i2} be the last link that meets F and E_{j2} the next link; $E_{i2} \neq E_{j2}$ since $\mathfrak{C}_2 > \mathfrak{C}$. By Lemmas VII 1.4 and VII 1.9 of [7], p. 201 ff, there is a cycle Γ^n mod M on F such that $\partial\Gamma^n \sim \gamma^{n-1}$ on M and $\Gamma^n \nsim 0 \bmod S-E_{i2}$ on F. Hence there exists a cocycle γ_n in E_{i2} such that $\gamma_n \cdot \Gamma^n = 1$ and $\gamma_n \nsim 0$ in E_{i1}. There exists a relation

$$a\gamma_n \sim bZ_n^i \quad \text{in} \quad E_{i1}, \tag{4.7a}$$

in which neither a nor b is zero. And by hypothesis there is a relation

$$\delta c^{n-1} = Z_n^i - Z_n^j \tag{4.7b}$$

(6) This apparent added restriction on F, not stated in 3.4s, may be stated here, since the existence of a different F satisfying the other conditions (and possibly not in Q) would imply S an orientable n-gcm, a fortiori satisfying 3.4s (cf. [7], p. 208, 2.19).

where c^{n-1} is in an open set $U \Subset E_{i1} \cup E_{j1}$. The portion of c^{n-1} on $E_{i2} \cup F$ is a chain c_1^{n-1} such that $\partial c_1^{n-1} = Z_n^i - Z_n$, where Z_n is in $E_{i1}-F$. And from the implied cohomology $Z_n^i \sim Z_n$ in E_{i2} and relation (4.7a) follows $a\gamma_n - bZ_n$ in E_{i1}. However, this implies $(a\gamma_n - bZ_n). \Gamma^n = 0$ ([7], p. 164, 18.24), and since $\gamma_n \cdot \Gamma^n = 1$, this in turn implies $Z_n \cdot \Gamma^n \neq 0$, which is impossible since Z_n is in $S-F$ and Γ^n is on F. From this we conclude that $F-M$ must be open in S.

5. Openness of an n-gm imbedded in an n-gm. Another application of Theorem 3.5 settles a question that arises below, namely, whether an n-gm imbedded in an n-gm S is open in S.

5.1. LEMMA. *If S_1 is the homeomorph of a locally compact space in a Hausdorff space S, and $x\varepsilon S_1 \cap \overline{S-S_1}$, then $x\varepsilon \bar{S}_1 \cap \overline{S-\bar{S}_1}$.*

Proof. Let $S_1 = f(S_2)$ where S_2 is a locally compact space and f a homeomorphism, and let $x = f(y)$, $y\varepsilon S_2$. As S_2 is locally compact, there is an open subset W of S_2 such that $y\varepsilon W$ and $\bar{W}$ is compact. Then $V = f(W)$ is open rel. S_1, $\bar{V} = f(\bar{W})$ is compact and $x\varepsilon V$. Also, $\bar{V}$ is closed in S. Let U be an open subset of S such that $U \cap S_1 \subset V$. Then it follows easily that $U \cap S_1 = U \cap \bar{S}_1$. The latter relation implies $U - U \cap S_1 = U - U \cap \bar{S}_1$; *i. e.*, $U - S_1 = U - \bar{S}_1$. Hence "x is a limit point of $S-S_1$" is equivalent to "x is a limit point of $S-\bar{S}_1$".

5.2. THEOREM. *Let S be a space satisfying axioms* (1)-(4), (6) *of* $\sum_n$. *Then axiom* (5) *of* Δ_n *is equivalent to the following assertion: If $S_1 = f(S_2)$ where $S_1 \subset S$, S_2 is a locally compact space and f a homeomorphism, and $y\varepsilon S_2$ such that $p_n(S_2, y) > 0$, then $x = f(y)$ is not a limit point of $S-S_1$.*

Proof. Suppose S satisfies axiom (5) of Δ_n. Then by Theorem 3.5, S has property 3.3. Now if x were a limit point of $S-S_1$, *i. e.*, $x\varepsilon S_1 \cap \overline{S-S_1}$, then by Lemma 5.1, $x\varepsilon\bar{S}_1 \cap \overline{S-\bar{S}_1}$. Hence by property 3.3, $p_n(\bar{S}_1, x) = 0$. But this would imply $p_n(S_1, x) = 0$, in contradiction to $p_n(S_1, x) = p_n(S_2, y) > 0$.

Conversely, to show that the assertion of the theorem implies axiom (5), it is sufficient by Theorem 3.5 to show that S has property 3.3. If A is a closed subset of S, then A is itself locally compact, and if $x\varepsilon A \cap \overline{S-A}$ it is necessary that $p_n(A, x) = 0$ since the contrary would imply, according to the assertion of the theorem, that $x \notin \overline{S-A}$.

5.3. COROLLARY. *If S is a locally orientable n-gm and S_1 the homeomorph in S of a locally compact space S_2 such that $p_n(S_2, y) > 0$ for all $y\varepsilon S_2$, then S_1 is open in S.*

5.4. COROLLARY. *The homeomorph in a locally orientable n-gm S of an n-gm is an open subset of S.*

Remark. It is clear that to solve the problem referred to at the end of § 2 above, it is sufficient to prove that $\sum_n$ implies any one of the properties 3.1-3.4s, or the properties embodied in Theorems 2.3, 2.4, 4.1, 4.3.

6. Application to orientability. In this section we study the orientability (in the large) of the locally orientable n-gm. Following a procedure similar to that of Poincaré [6], we obtain a characterization of orientability in terms of the local n-gms. An incidental result is a new derivation of the Begle condition cited above, as well as extension thereof to the non-compact case.

Basic is the following lemma, an immediate consequence of Lemma 2.1 and Theorem 2.2.

6.1. Lemma. *If S is an orientable n-gm with fundamental cycle* (see [7], p. 250) *Γ^n and S_1 is a connected open subset of S, then S_1 is an orientable n-gm whose fundamental cycle may be taken as the portion of Γ^n on S_1.*

6.2. In what follows, the assignment of a fundamental cycle Γ^n to an orientable n-gm S will be called *orienting* S; Γ^n may also be called the *orientation* of S. And if S and S_1 are related as in 6.1, then the orientation of the open set S_1 as assigned therein will be called the *orientation of S_1 induced* by Γ^n.

6.3. Lemma. *If S is an orientable n-gm and S_1, S_2 are intersecting connected open subsets of S, the former with orientation γ_1^n (arbitrarily assigned), then $S_1 \cup S_2$ can be assigned an orientation γ^n such that γ_1^n is the orientation of S_1 induced by γ^n. The orientation γ^n is independent of the orientation of S.*

Proof. By Lemma 6.1, there is an orientation Γ^n of $S_1 \cup S_2$ induced by the orientation of S. Then $a\gamma_1^n \sim b\Gamma^n \bmod S - S_1$, $a \neq 0 \neq b$. Let $S_1 \cup S_2$ be assigned the orientation $\gamma^n = (b/a)\Gamma^n$. If Γ_1^n were a different orientation of S, then there would exist a relation $\gamma_1^n \sim (b_1/a_1)\Gamma_1^n \bmod S - S_1$ implying $(b/a)\Gamma^n \sim (b_1/a_1)\Gamma_1^n \bmod S - S_1$, and therefore $(b/a)\Gamma^n = (b_1/a_1)\Gamma_1^n$ (cf. [7], p. 254-255).

6.4. In the symbols of the above proof, γ^n induces an orientation γ_2^n of S_2, and $\gamma_1^n \sim \gamma_2^n \bmod S - S_1 \cap S_2$. If in a space S, S_1 and S_2 are intersecting n-gms with respective orientations γ_1^n, γ_2^n such that $\gamma_1^n \sim \gamma_2^n \bmod S - S_1 \cap S_2$, then we shall say that S_1 and S_2 are *concurrently oriented*; or that their orientations are *concurrent*. And if S_1 and S_2 are intersecting n-gms in an n-gm S which can be assigned orientation in such a way as to render them concurrently oriented, we shall say that S_1 and S_2 *can be concurrently oriented*. Finally, the notion of "inducing" an orientation introduced above may be extended as follows: If S_1 has been assigned orientation γ_1^n and $S_1 \cup S_2$ can be assigned orientation γ^n so that $\gamma^n \sim \gamma_1^n \bmod S - S_1$, then the orientation γ_2^n of S_2 induced by γ^n will be called the *orientation of S_2 induced by γ_1^n*.

6.5. Lemma. *If S_1 and S_2 are intersecting orientable n-gms in an n-gm S, then in order that their orientations be concurrent it is necessary that their respective fundamental cocycles Z_n^1, Z_n^2 satisfy the cohomology $Z_n^1 \sim Z_n^2$ in $S_1 \cup S_2$.*

6.6. Lemma 6.5 is a corollary of the sufficiency of Lemma 6.7 below. That the condition of Lemma 6.5 is not sufficient is shown by the example of the projective plane, mod 3, in which there exist two overlapping 2-cells forming a Möbius band, which satisfy the condition but which cannot be concurrently oriented.

6.7. Lemma. *If S_1 and S_2 are intersecting orientable n-gms with orientations γ_1^n and γ_2^n, respectively, in an n-gm S, then a necessary and sufficient condition that there exist an orientation of $S_1 \cup S_2$ that induces the orientation γ_i^n of S_i, $i = 1, 2$, is that γ_1^n and γ_2^n be concurrent.*

Proof. That $S_1 \cap S_2$ is open in S follows from Corollary 5.4.

The necessity is trivial. The sufficiency is easily deduced from the relative Mayer-Vietoris sequence in terms of the groups $\mathfrak{H}^n(S_1)$, $\mathfrak{H}^n(S_2)$, etc. (see [5], p. 42 ff).

This Lemma is also a consequence of Théorème 8.1 of H. Cartan in the work [2] cited above.

6.8. If $\mathfrak{C}$ is any covering of an n-gm by open sets (we make no assumption about $\mathfrak{C}$ being locally finite or star-finite, and the same remark holds for the coverings $\mathfrak{C}_1$ and $\mathfrak{C}_2$ below, also), then there exist refinements $\mathfrak{C}_1$ and $\mathfrak{C}_2$ of $\mathfrak{C}$ similar to the coverings $\mathfrak{C}_1, \mathfrak{C}_2$ introduced in § 4, except that now the properties cited therein are not limited to the elements of $\mathfrak{C}_1$ and $\mathfrak{C}_2$ that meet some subset of S (such as P or Q). We shall prove the following theorem (proved by Begle, loc. cit., for the compact case):

6.9. Theorem. *An n-gm S is orientable if and only if for each covering $\mathfrak{C}$ of S by open sets there exist coverings $\mathfrak{C}_1, \mathfrak{C}_2$ and cocycles Z_n^i as defined above, such that for any choice of a canonical pair U, V in the intersection of elements of $\mathfrak{C}_2$ (as in § 4), the ratios a_i/a_j are all 1.*

Proof. For the necessity, suppose Γ^n is an orientation of S. Then $\mathfrak{C}_1$ and $\mathfrak{C}_2$ may be taken as identical, each element of $\mathfrak{C}_1$ being an n-gm $\mathfrak{C}_i$ whose orientation is induced by Γ^n as in Lemma 6.1, and whose corresponding cocycle Z_n^i is the fundamental cocycle of $\mathfrak{C}_i$; that $a_i/a_j = 1$ in all cases follows from Lemmas 6.5 and 6.7.

To prove the sufficiency, we note first that by Theorem 4.1, S is locally orientable. Hence if $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are given as in the hypothesis, we may assume all elements of $\mathfrak{E}_1$ and $\mathfrak{E}_2$ to be orientable n-gms. (Taking $\mathfrak{E}$ with elements orientable n-gms, each component of an element of $\mathfrak{E}_i$, $i = 1, 2$, is an orientable n-gm as in Lemma 6.1. Hence $\mathfrak{E}_1$ and $\mathfrak{E}_2$ may be replaced by new coverings having these components as elements). The orientations assigned to these elements will be determined as follows: Using the symbols of § 4, if $(x_i; E_{i1}, E_i; Z_n^i)$ is any one of the canonical triads such that E_{i1} and E_{i2} are corresponding elements of $\mathfrak{E}_1$ and $\mathfrak{E}$, respectively, then we select a $\gamma_{i1}^n \bmod S - E_{i1}$ such that $Z_n^1 \cdot \gamma_i^n = 1$ for the orientation of E_{i1}, and then orient $\mathfrak{E}_{i2}$ concurrently with orientation γ_{i2}^n. If $(x_j; E_{j1}, E_{j2}; Z_n^j)$ is such that $E_{i2} \cap E_{j2} \neq 0$, then $\gamma_{i2}^n \sim \gamma_{j2}^n \bmod S - E_{i2} \cap E_{j2}$. For suppose this were not the case. Then there would be a compact cocycle γ_n in $E_{i2} \cap E_{j2}$ such that $\gamma_n \cdot (\gamma_{i2}^n - \gamma_{j2}^n) \neq 0$, and since γ_n must lie in a finite number of components of $E_{i2} \cap E_{j2}$, we may assume it to lie in one such component, C. Select $(y; U, V; Z_n)$ in C with $U = V = C$ and orient C concurrently with E_{j1} (Lemma 6.1). By hypothesis there exist relations as in § 4 with $a_i/a_j = 1$, and $Z_n \cdot \gamma_{i1}^n = Z_n \cdot \gamma_{i2}^n = Z_n \cdot \gamma_{j2}^n = a$ $(= a_i = a_j \neq 0)$. But since C is an orientable n-gm, there must exist a cohomology $b\gamma_n \sim cZ_n$ in C, $b \neq 0 \neq c$. Hence $\gamma_n \sim \frac{c}{b} Z_n$ in C, implying $\frac{c}{b} Z_n \cdot (\gamma_{i2}^n - \gamma_{j2}^n) \neq 0$, which is impossible since $Z_n \cdot \gamma_{i2}^n = Z_n \cdot \gamma_{j2}^n$.

By Lemma 6.7 there exists an orientation γ_{ij}^n of $E_{i2} \cup E_{j2}$ such that $\gamma_{ij}^n \sim \gamma_{i2}^n \bmod S - E_{i2}$ and $\gamma_{ij}^n \sim \gamma_{j2}^n \bmod S - E_{j2}$. Commencing with a fixed E_{i2}, then, we may complete the orientation of any connected finite union of sets E_{i2} by induction. At each stage of the induction the union of all sets E_{i2} already selected is an n-gm U with orientation γ^n such that for any E_{i2} in U, $\gamma^n \sim \gamma_{i2}^n \bmod S - E_{i2}$. If $E_{k2} \varepsilon \mathfrak{E}_2$ is not in U, but intersects U, we can show $\gamma^n \sim \gamma_{k2} \bmod S - U \cap E_{k2}$ by selecting E_{i2} in U such that $E_{i2} \cap E_{j2} \neq 0$ and noting that $\gamma^n \sim \gamma_{i2}^n \sim \gamma_{k2}^n \bmod S - U_{i2} \cap E_{k2}$.

Now S, being locally compact, is the union of such finite unions of sets E_{i2}, and it is easy to see that if $U_1 \subset U_2$ are two such, then their orientations as defined above are concurrent. We may conclude, then, that those orientations determine a non-zero element in the inverse limit which defines the n-dimensional infinite homology group of S.

6.10. Definitions. If $\mathfrak{E}$ is a covering of a locally orientable n-gm S by n-gms E_i such that each $\mathrm{St}(E_i, \mathfrak{E})$ lies in an orientable n-gm, then the orientation γ_i^n of a single selected element E_i of $\mathfrak{E}$ will be called an *indicatrix* of S determined by E_i and γ_i^n. A finite sequence

(6.10a) $$E_i, E_{i(1)}, \ldots, E_{i(j)}, \ldots, E_{i(m)}, E_i$$

of elements of $\mathfrak{E}$ with identical initial and terminal elements and such that consecutive elements of the sequence intersect is called a *closed chain of* $\mathfrak{E}$; the elements of the sequence are called *links* of the closed chain. If E_i is the element of $\mathfrak{E}$ assigned the indicatrix γ_i^n, then in a closed chain (6.10a) γ_i^n *induces* an orientation $\gamma_{i(j)}^n$ of each $E_{i(j)}$ as follows: Since $\mathrm{St}(E_i, \mathfrak{E})$ lies in an n-gm S_1, γ_i^n induces a concurrent orientation $\gamma_{i(1)}^n$ of $E_{i(1)}$ (cf. Lemma 6.3 and § 6.4); then $\gamma_{i(1)}^n$ induces a concurrent orientation γ_{i2}^n of the next link $E_{i(2)}^{(n)}$ of the chain; and so on. In like manner, $\gamma_{i(m)}^n$ induces an orientation Z_n^i of E_i which will be called the orientation *of E_i induced by the closed chain* (6.10a) *and the indicatrix* γ_i^n.

6.11. Theorem. *In order that a locally orientable n-gm S should be orientable it is necessary and sufficient that for arbitrary covering $\mathfrak{U}$ of S by open sets there exist a refinement $\mathfrak{E}$ of $\mathfrak{U}$ whose elements are n-gms and an indicatrix of S determined by a special element E_i of $\mathfrak{E}$ and orientation γ_i^n of E_i, such that the orientation of E_i induced by all closed chains of $\mathfrak{E}$ is identical with γ_i^n.*

Proof. As for the necessity, if Γ^n is an orientation of S, let $\mathfrak{E}$ be a refinement of $\mathfrak{U}$ whose elements are n-gms and for a selected $E_i \varepsilon \mathfrak{E}$ let γ_i^n be the orientation of E_i induced by Γ^n. Then γ_i^n is the required indicatrix and the orientations induced by γ_i^n for the links of closed chains (6.10a) as defined in 6.10 are identical with those induced by Γ^n.

To prove the sufficiency, let $\mathfrak{U}$ be a covering of S such that for each $U \varepsilon \mathfrak{U}$, $\mathrm{St}(U, \mathfrak{U})$ lies in an orientable n-gm of S. Then with $\mathfrak{E}$ as given in the hypothesis, let E be an arbitrary element of $\mathfrak{E}$, and (6.10a) a closed chain in which E occurs as $E_{i(j)}$; as S is connected, such chains must exist ([7], p. 34, 12.5). Let $\gamma_{i(j)}^n$, be the orientation of E induced by γ_i^n as in 6.10. Then this orientation is the same for all closed chains of type (6.10a). For if

(6.11a) $$E_i, E_{k(1)}, \ldots, E_{k(h)}, \ldots, E_i$$

were another closed chain such that $E = E_{k(h)}$ and the corresponding induced orientation $\gamma_{k(h)}^n$ is not the same as $\gamma_{i(j)}^n$, then the sequence

(6.11b) $$E_i, E_{i(1)}, \ldots, E_{i(j)}, E_{k(h-1)}, \ldots, E_i$$

consisting of the beginning portion of (6.10a) from E_i to $E_{i(j)}$; and the beginning portion of (6.11a) *in reverse order* from $E_{k(h)}$ $(= E_{i(j)})$ back to E_i, is a closed chain in which the orientation of E_i induced by the chain is not identical with the indicatrix γ_i^n.

To see this, let γ^n be an orientation of $E \cup E_{k(h-1)}$ such that $\gamma^n \sim \gamma_{k(h)}^n \bmod S - E$ (Lemma 6.3). There exists a homology, $a\gamma_{i(j)}^n \sim b\gamma^n \bmod S - E$, $a \neq 0 \neq b$, $a \neq b$. As $\gamma_{i(j)}^n$ is the orientation of E induced by the chain (6.11b), $(b/a)\gamma^n$ is the orientation of $E_{k(h-1)}$ induced by the chain (6.11b); and evidently $(b/a)\gamma^n \sim (b/a)\gamma_{k(h-1)}^n \bmod S - E_{k(h-1)}$, where

$\gamma^n_{k(h-1)}$ is the orientation of $E_{k(h-1)}$ induced by the chain (6.11a). Continuing this process, it is shown that the orientation of E_i induced by (6.11b) is $(b/a)\gamma_i^n$, which is not γ_i^n since $a \neq b$.

This contradiction of the hypothesis shows that every $E\varepsilon\mathfrak{E}$ receives a unique orientation as a result of the assignment of the indicatrix to E_i. Now if $E_j, E_k \varepsilon \mathfrak{E}$ with orientations γ_j^n, γ_k^n so determined, then $\gamma_j^n \sim \gamma_k^n \bmod S - E_j \frown E_k$. For if $E_j \frown E_k \neq 0$ there is a closed chain in which E_j and E_k are consecutive links and in which the required homology holds by definition.

The existence of an orientation Γ^n of S can now be established by applying Theorem 6.9, with $\mathfrak{E} = \mathfrak{E}_1 = \mathfrak{E}_2$.

6.12. The completion of the above proof on the basis of Theorem 6.9 can be avoided by an induction argument such as was used in proving the latter theorem. This observation allows of avoiding the "arbitrarily small" element imposed by the injection of the covering $\mathfrak{U}$ in Theorem 6.11, and hence one can prove:

6.13. THEOREM. *If $\mathfrak{E}$ is a covering of an n-gm S by n-gms such that every* St$(E, \mathfrak{E})$, $E\varepsilon\mathfrak{E}$, *is orientable, and such that there exists an indicatrix of S determined by a special $E_i \varepsilon \mathfrak{E}$ and orientation γ_i^n of E_i such that the orientation of E_i induced by all closed chains of $\mathfrak{E}$ is identical with γ_i^n, then S has an orientation Γ^n such that $\Gamma^n \sim \gamma_i^n \bmod S - E_i$.*

Proof. As in the sufficiency proof of Theorem 6.11, it may be shown that each $E_j \varepsilon \mathfrak{E}$ receives a unique orientation γ_j^n, such that if $E_j, E_k \varepsilon \mathfrak{E}$ with orientations γ_j^n, γ_k^n, respectively, thus determined, then $\gamma_j^n \sim \gamma_k^n \bmod S - E_j \frown E_k$. Hence, applying Lemma 6.7 we may start an induction proof such as was used in Theorem 6.9. At the general stage of the induction we have, as before, a finite union U of n-gms which is itself an n-gm with orientation γ^n, say, such that for any $E_j \varepsilon \mathfrak{E}$ in U, $\gamma^n \sim \gamma_j^n \bmod S - E_j$. Let E_k be an element of $\mathfrak{E}$ meeting U but not a subset of U (for instance, since S is connected there exists $x \varepsilon \overline{U} \frown (S-U)$, and there exists an E_k such that $x \varepsilon E_k$).

We assert that $\gamma^n \sim \gamma_k^n \bmod S - U \frown E_k$. For suppose not. Then there exists in $U \frown E_k$ a cocycle Z_n such that

$$Z_n(\gamma^n - \gamma_k^n) = a \neq 0, \tag{6.13a}$$

and we may assume Z_n in a single component, C, of $U \frown E_k$. Let us orient C concurrently with γ^n, and select an $E_j \subset U$ such that $E_j \frown C \neq 0$ and γ_n^j in $E_j \frown C$ such that $\gamma_n^j \cdot \gamma_j^n = \gamma_n^j \cdot \gamma^n = 1$. Then there exists a relation $aZ_n - b\gamma_n^j \sim 0$ in C, $a \neq 0 \neq b$, and $Z_n \cdot \gamma^n = (b/a)\gamma_n^j \cdot \gamma^n = b/a \neq 0$. Now γ_j^n and γ_k^n are orientations of consecutive links in some closed chain of $\mathfrak{E}$, and hence $\gamma_j^n \sim \gamma_k^n \bmod S - E_j^n \frown E_k^n$ and therefore, since γ_n^j is in E_k, $\gamma_n^j \cdot \gamma_k^n = 1$. But then $Z_n \cdot \gamma_k^n = (b/a)\gamma_n^j \cdot \gamma_k^n = b/a$; and, finally, $Z_n \cdot (\gamma^n - \gamma_k^n) = 0$. But this contradicts (6.13a).

We conclude, then, that $\gamma^n \sim \gamma_k^n \bmod S - U \frown E_k$ and that the proof can then be concluded in the same manner as the proof of Theorem 6.9.

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