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EXTENSION OF AN EFFECTIVELY GENERATED CLASS OF FUNCTIONS BY ENUMERATION

BY

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1. In [1] Ackermann used double recursion to define a function which cannot be defined using only primitive recursion and explicit definition with 0 and $x+1$ as initial functions. Continuing from this result, Péter in a series of papers since 1935 has discussed a succession of types of recursion, producing larger and larger classes of functions (cf. [12]). Given a recursion defining a function φ, the n-tuples of natural numbers can be well-ordered so that the value of φ for any n-tuple depends by the recursion on values of φ only for preceding n-tuples. Péter has remarked how her successively stronger types of recursion are associated with increasing order-types of these well-orderings, and the same feature appears in the non-primitive recursions employed in Ackermann's consistency proof [2] for number theory. However this remark does not fully explain her hierarchy. For as Routledge [15] and Myhill [10] have observed, each general recursive function is definable using besides primitive recursion only a recursion over a suitable general recursive well-ordering of the natural numbers of order-type ω. In Myhill's treatement this well-ordering is even primitive recursive. Péter's hierarchy is kept from thus collapsing by her using actually only certain particular natural well-orderings of n-tuples for which no general criteria are given.

So a problem remains whether the ordinal numbers can be used to give a satisfactory classification of the general recursive functions into a hierarchy under some general principle (1). This might be by supplying criteria for the well-orderings of n-tuples above, or by using some other method for connecting ordinals with recursive number-theoretic functions.

(1) Such a classification does exist for general recursive functionals with one-place number-theoretic functions as argumente. To a general recursive $\varphi(\alpha)$, we can associate the ordinal $|S_1^R| < \omega_1$ of the set S_1^R of [7], § 26, (F) (or [8], XXII) for any primitive (or general) recursive ψ, R such that $\varphi(\alpha) = \psi\left(\bar{u}\left(\mu x R(\bar{\alpha}(x))\right)\right)$; such ψ, R exist by the normal form theorem ([5], Theorem IX*, p. 292, for $l = m_1 = 1$ and $n = 0$, with [6], footnote 2). The author has a proof, by a modification of the construction for [8], XXIV, that, to any ordinal $\xi < \omega_1$, there is a general recursive $\varphi(\alpha)$ for which any such associated ordinal $|S_1^R|$ is $\geqslant \xi$.

Another method for connecting ordinals with recursive number-theoretic functions is proposed in Hilbert's theory of recursions using increasing types of variables [4], which has been taken up again in recent work of Péter [13], [14]. Much remains to be investigated about this.

In the present note we shall propose still another method of associating ordinals with recursive functions, which appears to be worth investigating (²).

2. Given any countably infinite class C of number-theoretic functions, Cantor's diagonal method leads to number-theoretic functions outside C. We ask now whether it is possible to establish a hierarchy of classes of functions, starting from a given class as lowest, by repeated uses of diagonalization in association with ordinal numbers.

Suppose C contains all the constant one-place functions; then by a single diagonalization, followed by simple operations under which we shall wish our classes to be closed, we can obtain an arbitrary one-place function $\chi(a)$. For let the one-place functions of C be enumerated (allowing repetitions) as $\varphi_b(a)$ $(b = 0, 1, 2, \ldots)$ so that $\varphi_{2b}(a) = \chi(b)$, and diagonalize to obtain $\varphi(a) = \varphi_a(a)+1$. Then $\chi(a) = \varphi(2a)-1$. This example shows that the diagonalization (or the enumeration preceding it, which is the crucial step) must be controlled, if we are to get a non-trivial hierarchy based on the number of diagonalizations (or enumerations) required to define a function.

Now the cases that interest us are when C is a special class of recursive functions, such as the primitive recursion functions, or the Csillag-Kalmár elementary functions (³), or the functions primitive recursive (or elementary) in a given function θ. In all these cases C is given effectively (or effectively from θ) as the class of the functions generated from certain initial functions by repeated application of certain operations. This manner of generation makes certain enumerations of the one-place functions of C immediate. The enumeration (or diagonalization)

The proof we had uses a modification of the construction for [8], XXIV. The following is simpler: For any Q such that $(\alpha)(Ex)Q(\overline{\alpha}(x))$, the functional $\varphi(\alpha) = \overline{\alpha}\big(\mu x Q(\overline{\alpha}(x))\big)$ has no associated ordinal $<|S_1^Q|$. Using the recursion theorem [5], p. 352, pick a partial recursive predicate Q such that $Q(y, v) \simeq v = 1$ if $y = 1$, $\simeq \mathrm{lh}(v) > 0 \,\&\, Q\big((y)_0, \Pi_{i<v} p_i^{(v)_i+1}\big)$ if $y = 2^{(y)_0} \neq 1$, $\simeq \mathrm{lh}(v) > 0 \,\&\, Q\Big(\{(y)_2\}([(v)_0 \dot{-} 1]_0), \Pi_{i<v} p_i^{(v)_i+1}\Big)$ if $y = 3 \cdot 5^{(y)_2}$. Then for each $y \in O$, $\lambda v Q(y, v)$ is completely defined, $(\alpha)(Ex)Q\big(y, \overline{\alpha}(x)\big)$ and $|S_1^{Q,y}| \geqslant |y|$.

(²) This method essentially was proposed to the Seminar on the Foundations of Mathematics at the University of Wisconsin in April 1953. Paul Axt answers some of the questions that arise concerning it in his thesis [3].

(³) Cf. [5], p. 285, where references to Kalmár's papers are given. Our notations not otherwise explained are taken from [5] (cf. bottom p. 538).

will be controlled by using only such an enumeration $\varphi_b(a)$. If, to the initial functions for the generation, we adjoin $\varphi(b, a) = \varphi_b(a)$ (whence $\varphi_a(a)+1$ can be generated), we get an enlarged class C' generated like C except for there being the additional initial function. This gives the idea which we shall use in its essentials to rise from one class C to another C', corresponding to the step from an ordinal α to its successor $\alpha+1$.

Corresponding to the passage from a fundamental sequence of ordinals $\alpha_0, \alpha_1, \alpha_2, \ldots$ to $\alpha = \lim_n \alpha_n$, we can similarly adjoin a function which combines the enumerations of the classes corresponding to $\alpha_0, \alpha_1, \alpha_2, \ldots$ However, just as the diagonalization or enumeration of a given class must be controlled, here the formation of fundamental sequences must be controlled. The control will be provided by an appropriate version of the Church-Kleene theory of constructive ordinals.

In the following sections, we shall work out the idea in detail for the case the initial class is the primitive recursive functions. The reader should then have no trouble in proceeding similarly from e. g. the elementary functions as the initial class.

3. We now assign numbers, to be called *indices (indices from Θ)* to the primitive recursive functions (functions primitive recursive in assumed functions Θ). An index of a primitive recursive function φ (a function φ primitive recursive in Θ) will be obtained by writing in code form via prime-factor representation the analysis of a primitive recursive description of φ (primitive recursive derivation of φ from Θ) (⁴). Thus an index of a function will reflect the generation of it by a series of applications of listed schemata, while a Gödel number of it reflects the linguistic form of a system E of equations defining it recursively (⁵).

Let $\langle a_0, \ldots, a_n\rangle = p_0^{a_0} \cdot \ldots \cdot p_n^{a_n} (= 1$ when $n = -1$, i. e. $\langle\,\rangle = 1)$.

For a fixed list of $l \geqslant 0$ assumed functions $\theta_1(a_1, \ldots, a_{m_1}), \ldots, \theta_l(a_1, \ldots, a_{m_l})$, or briefly θ, a function φ primitive recursive in Θ can be introduced by one of the following schemata, where in (IV) and (V) $\psi, \chi_1, \ldots, \chi_m, \chi$ are functions previously introduced by applications of

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(⁴) Cf. [5], pp. 220, 224; the analysis is the explanation written at the right, e. g., in Examples 1 and 2, p. 221-222 (cf. p. 234).

(⁵) The Gödel numbers of general recursive functions, introduced by the author in 1936 (cf. [5], pp. 289, 292), were so-called because they are obtained by using Gödel's method or a modification of it to number the systems E. We could use now the subclass of Gödel numbers obtained by restricting the E's to those of a suitable special form (cf. [5], Lemma IIa, p. 267), but using the indices avoids the bother of arithmetizing a language. Indices can also be used for general and partial recursive functions, e. g. on the basis of six schemata ([5], pp. 279, 289, 330, 331); a treatment, including the normal form theorem, intended for [5] was left out to save space, but we plan to use such indices in future publications.

the schemata, and $m, n \geqslant 0$ except that $n \geqslant 1$ in (III) and (V). The corresponding indices are written at the right where for (IV) and (V) $g, h_1, \ldots, h_m, h$ are indices of $\psi, \chi_1, \ldots, \chi_m, \chi$ as previously introduced [6]:

$$(\;)\quad \varphi(a_1, \ldots, a_{m_i}) = \theta_i(a_1, \ldots, a_{m_i}) \qquad \langle 0, m, i\rangle.$$

$$(\mathrm{I})\quad \varphi(a) = a' = a+1 \qquad \langle 1, 1\rangle.$$

$$(\mathrm{II})\quad \varphi(a_1, \ldots, a_n) = q \qquad \langle 2, n, q\rangle.$$

$$(\mathrm{III})\quad \varphi(a_1, \ldots, a_n) = a_i \qquad \langle 3, n, i\rangle.$$

$$(\mathrm{IV})\quad \varphi(a_1, \ldots, a_n) = \psi(\chi_1(a_1, \ldots, a_n), \ldots, \chi_m(a_1, \ldots, a_n)) \qquad \langle 4, n, g, h_1, \ldots, h_m\rangle.$$

$$(\mathrm{V})\quad \begin{cases} \varphi(0, a_2, \ldots, a_n) = \psi(a_2, \ldots, a_n), \\ \varphi(a_1', a_2, \ldots, a_n) = \chi(a_1, \varphi(a_1, a_2, \ldots, a_n), a_2, \ldots, a_n) \end{cases} \qquad \langle 5, n, g, h\rangle.$$

For fixed $l, m_1, \ldots, m_l \geqslant 0$, we write $\mathrm{In}^{m_1, \ldots, m_l}(b)$ to say that b is an index of a function φ from functions Θ (as above). $\mathrm{In}^{m_1, \ldots, m_l}$ is primitive recursive, since it satisfies the course-of-values recursion

$$\begin{aligned} \mathrm{In}^{m_1, \ldots, m_l}(b) \equiv\ & b = \langle 0, m_1, 1\rangle \vee \ldots \vee b = \langle 0, m_l, l\rangle \vee b = \langle 1, 1\rangle \\ & \vee b = \langle 2, (b)_1, (b)_2\rangle \vee \{b = \langle 3, (b)_1, (b)_2\rangle \,\&\, 1 \leqslant (b)_2 \leqslant (b)_1\} \\ & \vee \Big\{b = \prod_{i<(b)_{2,1}+3} p_i^{(b)_i} \,\&\, (b)_0 = 4 \,\&\, \mathrm{In}^{m_1, \ldots, m_l}((b)_2) \,\&\, \\ & \qquad (i)_{2<i<(b)_{2,1}+3}[\mathrm{In}^{m_1, \ldots, m_l}((b)_i) \,\&\, (b)_{i,1} = (b)_1]\Big\} \\ & \vee b = \langle 5, (b)_1, (b)_2, (b)_3\rangle \,\&\, \mathrm{In}^{m_1, \ldots, m_l}((b)_2) \,\&\, \\ & \qquad (b)_{2,1}+1 = (b)_1 \,\&\, \mathrm{In}^{m_1, \ldots, m_l}((b)_3) \,\&\, (b)_{3,1} = (b)_1+1\}. \end{aligned}$$

4. If φ is a primitive recursive in functions $\theta_1(a_1, \ldots, a_{m_1}), \ldots, \theta_l(a_1, \ldots, a_{m_l})$, or briefly Θ, with index b, and Θ are primitive recursive in Ψ with respective indices $c_1, \ldots, c_l$, then φ is primitive recursive in Ψ with index $\mathrm{tr}^{m_1, \ldots, m_l}(b, c_1, \ldots, c_l)$ where $\mathrm{tr}^{m_1, \ldots, m_l}$ is the primitive recursive function defined thus:

(6) This agrees with basis B of [5], pp. 223, 238, except that we are letting (II) give all constant functions as initial functions. (In this paper we are using "initial function" relative to whatever class C is at the moment being generated, so for $C = \{$functions primitive recursive in $\theta\}$, the initial functions include both the "initial" and "assumed" functions of IM, pp. 219, 224.)

$$\begin{aligned} \mathrm{tr}^{m_1, \ldots, m_l}(b, c_1, \ldots, c_l) &= c_i \text{ if } b = \langle 0, m_i, i\rangle \qquad (i = 1, \ldots, l), \\ &= b \text{ if } \mathrm{In}^{m_1, \ldots, m_l}(b) \,\&\, 1 \leqslant (b)_0 \leqslant 3, \\ &= 2^4 \cdot 3^{(b)_1} \cdot \textstyle\prod_{1<i<(b)_{2,1}+3} p_i \exp \mathrm{tr}^{m_1, \ldots, m_l}((b)_i, c_1, \ldots, c_m) \\ &\qquad \text{if } \mathrm{In}^{m_1, \ldots, m_l}(b) \,\&\, (b)_0 = 4, \\ &= \langle 5, (b)_1, \mathrm{tr}^{m_1, \ldots, m_l}((b)_2, c_1, \ldots, c_m), \mathrm{tr}^{m_1, \ldots, m_l}((b)_3, c_1, \ldots, c_m)\rangle \\ &\qquad \text{if } \mathrm{In}^{m_1, \ldots, m_l}(b) \,\&\, (b)_0 = 5, \\ &= 0 \text{ otherwise.} \end{aligned}$$

If moreover Ψ are functions of $k_1, \ldots, k_j$ variables respectively, $\mathrm{In}^{k_1, \ldots, k_j}(\mathrm{tr}^{m_1, \ldots, m_l}(b, c_1, \ldots, c_l))$ only when $\mathrm{In}^{m_1, \ldots, m_l}(b)$.

5. In the case that $\mathrm{In}^{m_1, \ldots, m_l}(b)$, we write $\mathrm{pr}_b^{\Theta}(a_1, \ldots, a_n)$ for the function primitive recursive in Θ with index b, where $n = (b)_1$ by the definition of "index".

Let

$$\mathrm{pr}^{\Theta}(b, a) = \begin{cases} \mathrm{pr}_b^{\Theta}((a)_0, \ldots, (a)_{(b)_1 - 1}) & \text{if } \mathrm{In}^{m_1, \ldots, m_l}(b), \\ 0 \text{ otherwise.} \end{cases}$$

Now for each fixed $n \geqslant 0$,

$$\mathrm{pr}^{\Theta}(0, \langle a_1, \ldots, a_n\rangle), \mathrm{pr}^{\Theta}(1, \langle a_1, \ldots, a_n\rangle), \mathrm{pr}^{\Theta}(2, \langle a_1, \ldots, a_n\rangle), \ldots$$

is an enumeration with repetitions of the n-place functions primitive recursive in Θ. For if $\varphi(a_1, \ldots, a_n)$ is primitive recursive in Θ, it has an index b from Θ, and $\varphi(a_1, \ldots, a_n) = \mathrm{pr}_b^{\Theta}(a_1, \ldots, a_n) = \mathrm{pr}^{\Theta}(b, \langle a_1, \ldots, a_n\rangle)$. Conversely, for each b, $\mathrm{pr}^{\Theta}(b, \langle a_1, \ldots, a_n\rangle)$ is primitive recursive in Θ, since

$$\begin{aligned} \mathrm{pr}^{\Theta}(b, \langle a_1, \ldots, a_n\rangle) &= \mathrm{pr}_b^{\Theta}(a_1, \ldots, a_{(b)_1}) && \text{if } \mathrm{In}^{m_1, \ldots, m_l}(b) \,\&\, (b)_1 \leqslant n, \\ \mathrm{pr}^{\Theta}(b, \langle a_1, \ldots, a_n\rangle) &= \mathrm{pr}_b^{\Theta}(a_1, \ldots, a_n, 0, \ldots, 0) && \text{if } \mathrm{In}^{m_1, \ldots, m_l}(b) \,\&\, (b)_1 \geqslant n, \\ \mathrm{pr}^{\Theta}(b, \langle a_1, \ldots, a_n\rangle) &= 0 && \text{if } \overline{\mathrm{In}}^{m_1, \ldots, m_l}(b). \end{aligned}$$

The "enumerating function" $\mathrm{pr}^{\Theta}(b, a)$ is not primitive recursive in Θ. For if it were, so would be $\mathrm{pr}^{\Theta}(a, \langle a\rangle)+1$; but by Cantor's diagonal reasoning, the latter function is not in the enumeration $\mathrm{pr}^{\Theta}(0, \langle a\rangle)$, $\mathrm{pr}^{\Theta}(1, \langle a\rangle)$, $\mathrm{pr}^{\Theta}(2, \langle a\rangle), \ldots$ of the one-place functions primitive recursive in Θ.

On the other hand, for $i = 1, \ldots, l$, the function θ_i is primitive recursive in $\mathrm{pr}^{\Theta}(b, a)$, since $\theta_i(a_1, \ldots, a_{m_i}) = \mathrm{pr}^{\Theta}(\langle 0, m_i, i\rangle, \langle a_1, \ldots, a_{m_i}\rangle)$.

6. Say that A, B, C are each a number-theoretic function, predicate or set. The relation "A is primitive recursive in B" is reflexive and transitive. So "A is primitive recursive in B, and B is primitive recursive in A" is reflexive, transitive and symmetric, and hence divides the functions, predicates and sets into equivalence classes, which we call *primitive recursive degrees*, in analogy to the *degrees of recursive unsolvability* (or *general recursive degrees*) of Kleene and Post [9] (cf. p. 381).

The definitions of $\mathbf{a} \leqslant \mathbf{b}$, $\mathbf{a} < \mathbf{b}$, etc., $\mathbf{0}, \mathbf{a} \cup \mathbf{b}$, and the formulas (1)-(9) of [9], 1.2 and 1.3, now hold substituting throughout primitive recursiveness for general recursiveness.

In the case ($l = 1$, $m_1 = 2$) that Θ is a single two-place function $\theta(b, a)$, we shall sometimes write $\mathrm{pr}^{\Theta}(b, a)$ simply $\theta'(b, a)$.

Suppose $\psi(a_1, a_2)$ is primitive recursive in $\chi(a_1, a_2)$ with index c. Then by § 4, $\psi'(b, a) = \chi'(\mathrm{tr}^2(b, c), a)$. Thus, when ψ is primitive recursive in χ, ψ' will be primitive recursive in χ'.

Applying this remark to the case ψ, χ are first θ_1, θ_2 and then θ_2, θ_1 for two two-place functions θ_1, θ_2 of the same (primitive recursive) degree, we have that the degree, call it $\mathbf{a}'$, of θ' is determined by the degree $\mathbf{a}$ of θ; i. e. we can construe $'$ as an operation on degrees. Then the remark gives

$$\mathbf{a} \leqslant \mathbf{b} \to \mathbf{a}' \leqslant \mathbf{b}', \tag{10}$$

and the conclusion of § 5, for θ as the Θ, can be stated

$$\mathbf{a} < \mathbf{a}'. \tag{11}$$

Thus we have an operation $'$ which raises primitive recursive degrees analogous to the operation $'$ of Kleene and Post [9], 1.4 which raises general recursive degrees, and their formulas (1)-(12) all hold. How much further the theory of primitive recursive degrees can be developed to parallel that of general recursive degrees has been investigated by Axt [3].

7. We are now in a position to set up a hierarchy of two-place number-theoretic functions h_y with ascending primitive recursive degrees, analogous to the hierarchy H_y of one-place predicates of Kleene [6], § 6.

For the hierarchy H_y the control over the formation of fundamental sequences was provided by letting y range, not over Cantor's first and second number classes, but over the notations y, of the set O (partially ordered by $<_O$), for ordinals $|y|$ in the system S_3 formed under the restriction to general recursive fundamental sequences (7). That control was necessary in order that the hierarchy not collapse by a predicate of arbitrary degree being definable at the ω level, as is shown by [8], XIII, p. 199.

(7) Cf. [7], § 20, or [8], p. 199-200, where further references are given.

For the present hierarchy based on primitive recursiveness, it would be out of keeping to allow more than primitive recursive fundamental sequences. So here we are primarily interested in the case the $O, <_O$ and $|\;|$ (called $O', <'_O$ and $|\;|'$ in §§ 10, 11) differ from those of S_3 by being formed under the restriction to primitive recursive fundamental sequences. That this restriction is necessary, if the hierarchy is not to collapse by a function of arbitrary primitive recursive degree for a general recursive predicate being definable at the ω level, has been shown by Axt [3].

In §§ 10, 11 we shall compare the two systems of notation for ordinals, differing by the use of general, or only primitive, recursive fundamental sequences. The intervening material and § 12 actually read correctly for either system, though principally we intend the latter. In particular, when $3 \cdot 5^z$ is a notation for a limit ordinal $\alpha = \lim_n \alpha_n$, then z_n or $[z]_n$ is the notation for α_n in the fundamental sequence of notations having $3 \cdot 5^z$ as limit notation.

We now define $h_y(b, a)$, for each $y \varepsilon O$, thus: 1. $h_1(b, a) = 0$. 2. If $y \varepsilon O$ and $y = 2^z \neq 1$, then $h_y(b, a) = \mathrm{pr}^{h_z}(b, a)$, or more briefly $h_y = h'_z$. 3. If $y \varepsilon O$ and $y = 3 \cdot 5^z$, then $h_y(b, a) = h_{\mathrm{A}}((b)_0, a)$ where $\mathrm{A} = z_{(b)_1}$.

The hierarchy can be relativized to any given number-theoretic function $\varkappa(b, a)$. If we parallel the relativization of H_y in [6], 6.5 (but cf. also 6.8), the definition is as follows: 1. $h_1^{\varkappa}(b, a) = \varkappa(b, a)$. 2. If $y \varepsilon O^{\varkappa}$ and $y = 2^z \neq 1$, then $h_y^{\varkappa}(b, a) = \mathrm{pr}^{\varkappa, h_z^{\varkappa}}(b, a)$. 3. If $y \varepsilon O^{\varkappa}$ and $y = 3 \cdot 5^z$, then $h_y^{\varkappa}(b, a) = h_{\mathrm{A}}^{\varkappa}((b)_0, a)$ where $\mathrm{A} = z_{(b)_1}^{\varkappa}$.

8. The (primitive recursive) degree of h_y we write $\mathbf{0}_{(y)}$.

In case 3 of the definition of h_y: For each n, $h_{z_n}(b, a) = h_y(\langle b, n \rangle, a)$, so h_{z_n} is primitive recursive in h_y.

Using this remark with (11), by an induction with cases corresponding to the clauses (§ 10, or [7], § 20) by which $y <_O z$ can hold,

$$y <_O z \to \mathbf{0}_{(y)} < \mathbf{0}_{(z)}. \tag{13a}$$

For any $y \varepsilon O$, define $C_y = \{$the functions primitive recursive in $h_y\}$ (8). Then by (13a)

$$y <_O z \to (C_y \subset C_z \; \& \; C_y \neq C_z). \tag{13b}$$

(8) We have departed from the heuristic account in § 2 in two inessential respects. To rise from $C = C_z$ to $C' = C_{2^z}$, we could (keeping to the idea of § 2) have adjoined $\mathrm{pr}^{h_z}(b, \langle a \rangle)$, which enumerates the one-place functions of C; however, in generating C the one-place functions are obtained interspersed among n-place functions for $n \neq 1$, and it seems more natural to adjoin $\mathrm{pr}^{h_z}(b, a)$, which via the contraction of $(a_1, \ldots, a_n)$ to $a = \langle a_1, \ldots, a_n \rangle$ enumerates all the functions $\varphi(a_1, \ldots, a_n)$ (n varying) of C in a manner which may be considered as immediate from the generation of C. The other departure is that, instead of at each step adjoining a new initial

9. Finally, we show that, for each $y \varepsilon O$, the function h_y, and hence all the functions C_y, are general recursive.

The function $\mathrm{pr}^\theta(b, a)$ comes from $\theta(b, a)$ by the double course-of-values recursion

$$
\begin{aligned}
(14)\qquad \mathrm{pr}^\theta(b, a) &= \theta\big((a)_0, (a)_1\big) \text{ if } \mathrm{In}^2(b) \ \&\ (b)_0 = 0, \\
&= (a)_0' \text{ if } \mathrm{In}^2(b) \ \&\ (b)_0 = 1, \\
&= (b)_2 \text{ if } \mathrm{In}^2(b) \ \&\ (b)_0 = 2, \\
&= (a)_{(b)_2 \dot{-} 1} \text{ if } \mathrm{In}^2(b) \ \&\ (b)_0 = 3, \\
&= \mathrm{pr}^\theta\big((b)_2, \textstyle\prod_{i<(b)_{2,1}} p_i^{\mathrm{pr}^\theta((b)_{i+3}, a)}\big) \text{ if } \mathrm{In}^2(b) \ \&\ (b)_0 = 4, \\
&= \mathrm{pr}^\theta\big((b)_2, \textstyle\prod_{i<(b)_1 \dot{-} 1} p_i^{(a)_{i+1}}\big) \text{ if } \mathrm{In}^2(b) \ \&\ (b)_0 = 5 \ \&\ (a)_0 = 0, \\
&= \mathrm{pr}^\theta\big((b)_3, 2^{(a)_0 \dot{-} 1} \cdot 3^{\mathrm{pr}^\theta(b, [a/2])} \cdot \textstyle\prod_{i<(b)_1 \dot{-} 1} p_{i+2}^{(a)_{i+1}}\big) \\
&\qquad\qquad \text{ if } \mathrm{In}^2(b) \ \&\ (b)_0 = 5 \ \&\ (a)_0 > 0, \\
&= 0 \text{ otherwise.}
\end{aligned}
$$

As this illustrates for $l = 1$, $m_1 = 2$, pr^Θ is double recursive uniformly in Θ (9).

Using the recursion theorem ([5], p. 352), choose a partial recursive function $h(y, b, a)$ such that

$$
(15)\qquad h(y, a, b) \simeq \begin{cases} \mathrm{pr}^{\lambda ba h((y)_0, b, a)}(b, a) & \text{if } \ y = 2^{(y)_0}, \\ h([(y)_2]_{(b)_1}, (b)_0, a) & \text{if } \ y = 3 \cdot 5^{(y)_2}, \\ 0 \quad \text{otherwise.} \end{cases}
$$

Now, for each $y \varepsilon O$, $h(y, b, a) = h_y(b, a)$.

10. Some properties (I)-(XXIII) of the predicates $a \varepsilon O$ and $a <_O b$ of the system S_3 of notation for ordinals are collected in [7], §§ 20-23. We add one more:

(XXIV) *If* $a <_O b$, *then* $2^a \leqslant_O b <_O 2^b$.

In adapting S_3 to use only primitive recursive fundamental sequences $\{y_n\}$, we can either continue using Gödel numbers, or use indices (§§ 3,5), in describing the fundamental sequences. We choose to use indices, which seems more in keeping with the restriction to primitive recursiveness.

The resulting modifications of the notions $a \varepsilon O$, $a <_O b$, $|a|$ of S_3 we distinguish in this and the next section by accents: $a \varepsilon O'$. $a <_O' b$, $|a|'$. Their definitions read as before (e. g. [7], § 20), except accents are inserted and $O3$ is replaced by: $O'3$. If $\mathrm{In}(y) \ \&\ (y)_1 = 1$ and, for each n, $y_n \varepsilon O'$ and $y_n <_O' y_{n+1}$, where $y_n = \mathrm{pr}(y, \langle n_O \rangle)$, then $3 \cdot 5^y \varepsilon O'$ and, for each n, $y^y <_O' 3 \cdot 5^y$ (and $|3 \cdot 5^y|' = \lim_n |y_n|'$).

Properties (I)-(XXIV) hold anew for the accented notions (including $C'(b)$, $a +_O' b$, $\sum_O'$). Here $a +_O' b$ is defined by reworking the definition of $a +_O b$ (see [7], p. 412-413) to make $d_{a,y}$ in case 3 an index of $\lambda n\ a +_O' \mathrm{pr}(y, \langle n \rangle)$, as follows.

First, we adapt S_n^m ([5], p. 342): *The function*

$$
\begin{aligned}
&\mathrm{sb}_n^m(z, y_1, \ldots, y_m) \\
&\qquad = \langle 4, n, z, \langle 2, n, y_1 \rangle, \ldots, \langle 2, n, y_m \rangle, \langle 3, n, 1 \rangle, \ldots, \langle 3, n, n \rangle \rangle
\end{aligned}
$$

is primitive recursive, and, when z *is an index of a primitive recursive function* $\varphi(y_1, \ldots, y_m, a_1, \ldots, a_n)$, *then, for each fixed* $y_1, \ldots, y_m$, *the number* $\mathrm{sb}_n^m(z, y_1, \ldots, y_m)$ *is an index of* $\lambda a_1 \ldots a_n \varphi(y_1, \ldots, y_m, a_1, \ldots, a_n)$.

Next, we adapt the recursion theorem ([5], p. 352): *To any primitive recursive function* $\psi(z, a_1, \ldots, a_n)$, *there is an index* e *of* $\lambda a_1 \ldots a_n \psi(e, a_1, \ldots, a_n)$. For, let d be an index of $\lambda b a_1 \ldots a_n \psi(\mathrm{sb}_n^1(b, b), a_1, \ldots, a_n)$, and take $e = \mathrm{sb}_n^1(d, d)$. (Both these adaptations hold likewise for functions primitive recursive in Θ with indices from Θ.)

Now, to define $+_O'$, let $\theta(z, a, y) = \langle 4, 1, z, \langle 2, 1, a \rangle, y \rangle$, so that (when $\mathrm{In}(y) \ \&\ (y)_1 = 1$) $\theta(e, a, y)$ will be an index of $\lambda n\ a +_O' \mathrm{pr}(y, \langle n \rangle)$ if e is one of $\lambda ab\ a +_O' b$. Let

$$
\psi(z, a, b) = \begin{cases} a \text{ if } b = 1 \ \&\ a \neq 0, \\ 2^{\psi(z, a, (b)_0)} \text{ if } b = 2^{(b)_0} \neq 1, \\ 3 \cdot 5^{\theta(z, a, (b)_2)} \text{ if } b = 3 \cdot 5^{(b)_2}, \\ 7 \text{ otherwise.} \end{cases}
$$

This is a course-of-values recursion, so $\psi(z, a, b)$ is primitive recursive. Pick an e for this $\psi(z, a, b)$ by the (new) recursion theorem, and let $a +_O b = \psi(e, a, b)$.

11. In this section we show that O' has notations for exactly the same ordinals (those $< \omega_1$) as O.

It has notations for as many ordinals: *There is a primitive recursive function* π *such that, for each* $c \varepsilon O$:

function for the generation of our class, we have for each y an initial function h_y with index $\langle 0, 2, 1 \rangle$ (the sole assumed function for the relative primitive recursiveness) which at each step we simply change (the former such functions not being lost since they are primitive recursive in the new one); the first class C_1 is the primitive recursive functions, but generated as the functions primitive recursive in $\lambda ba\ 0$.

(9) This for Θ empty constitutes a quite expeditious version of the diagonalization proof (Péter [11], § 2) of the existence of non-primitive double recursive functions.

(i) *If* $a \leqslant_O c$, *then* $\pi(a) \epsilon O'$ *and* $|\pi(a)|' = |a|$.

(ii) *If* $a <_O b \leqslant_O c$, *then* $\pi(a) <'_O \pi(b)$.

The proof of the two properties of π will be by induction on c over O. First, we treat the three cases (O1-O3) by which c can ϵO, in each case giving a definition of $\pi(c)$ that suffices for the case.

Case 1: $c = 1$. Define $\pi(c) = 1$. Using (II), we have (i) and (ii).

Case 2: $c = 2^y$ with $y \epsilon O$. Define $\pi(c) = 2^{\pi(y)}$. Using (V) and the hypothesis of the induction, we again have (i) and (ii).

Case 3: $c = 3\cdot 5^y$ and $c \epsilon O$. Using the hypothesis of the induction, $\pi(y_0), \pi(y_1), \pi(y_2), \ldots$ where $y_i = \Phi(y, i_O)$ is a sequence of members of O', ascending in the sense $<_{O'}$. Suppose p is a Gödel number of π. Let q be a Gödel number of $\lambda pci\Phi(p, \Phi((c_2, i_O))$. Then $s = S_1^2(q, p, c)$ is a Gödel number of the sequence $\pi(y_0), \pi(y_1), \pi(y_2), \ldots$; *i. e.*, for each i, $T(s, i, t)$ holds for a unique number $t = \tau(i)$, and $\pi(y_i) = U(\tau(i))$. Now (cf. [7], p. 415-416) let

$$\psi(s, 0) = 0_O = 1,$$

$$\psi(s, n+1) = \begin{cases} \pi(y_i) \text{ if } n = \bar{\tau}(i+1), \\ \psi(s, n) \text{ if } \overline{(Ei)}[n = \bar{\tau}(i+1)] \end{cases}$$

$$= \begin{cases} U((n_{\mathrm{lh}(n)\dot{-}1}\dot{-}1) \text{ if } \mathrm{Seq}(n) \,\&\, \mathrm{lh}(n) > 0 \,\&\, (i)_{i<\mathrm{lh}(n)} T(s, i, (n)_i\dot{-}1), \\ \psi(s, n) \text{ otherwise.} \end{cases}$$

Thus $\psi(s, 0), \psi(s, 1), \psi(s, 2), \ldots$ arises from $\pi(y_0), \pi(y_1), \pi(y_2), \ldots$ by first replacing each member by some succession of repetitions of itself and then prefixing some 1's. Next let $\varphi(s, n) = \psi(s, n) +'_O n_O$.

Now (a) for each n, $\varphi(s, n) \epsilon O'$ and $\varphi(s, n) <'_O \varphi(s, n+1)$. For $\psi(s, n) \leqslant'_O \psi(s, n+1)$ (since $\pi(y_i)$ is increasing), whence $\varphi(s, n) = \psi(s, n) +'_O n_O \leqslant'_O \psi(s, n+1) +'_O n_O$ (using (XXIV) n times) $<'_O \psi(s, n+1) +'_O (n+1)_O = \varphi(s, n+1)$.

Furthermore (b) to each i there is an n (namely $\bar{\tau}(i+2)$) such that $\pi(y_i) <'_O \varphi(s, n)$.

Vice versa (c) to each n there is an i such that $\varphi(s, n) <'_O \pi(y_i)$. For, $\psi(s, n) \leqslant'_O \pi(y_j)$ for some j, whence (using (XXIV) n times) $\varphi(s, n) <'_O \pi(y_{j+n+1})$.

The function φ is primitive recursive. Let e be an index of $\lambda sa\, \varphi(s, \mathrm{nat}(a))$ where $\mathrm{nat}(n_O) = n$ (cf. [7], p. 410). Let $d = \mathrm{sb}_1^1(e, s)$, so d is an index of $\varphi(s, n)$ as a function of n_O. Define $\pi(3\cdot 5^y) = 3\cdot 5^d$.

To prove (i), assume $a \leqslant 3\cdot 5^y$.

Subcase 1: $a = 3\cdot 5^y$. By (a) and the choice of d, $3\cdot 5^d \epsilon O'$, i. e. $\pi(a) \epsilon O'$. Also using (b), (c), and (i) of the hypothesis of the induction,

$$|\pi(a)|' = \lim_n |\varphi(s, n)|' = \lim_i |\pi(y_i)|' = \lim_i |y_i| = |a|.$$

Subcase 2: $a < 3\cdot 5^y$. Use (VI) and the hypothesis of the induction (i).

To prove (ii), assume $a <_O b \leqslant_O 3\cdot 5^y$.

Subcase 1': $b = 3\cdot 5^y$. For some i, $a \leqslant_O y_i$, so $a <_O y_{i+1} \leqslant_O y_{i+1}$, and by (ii) of the hypothesis of the induction, $\pi(a) <'_O \pi(y_{i+1})$. But using (b), for some n, $\pi(y_{i+1}) <'_O \varphi(s, n) <'_O 3\cdot 5^d = \pi(b)$. Thus $\pi(a) <'_O \pi(b)$.

Subcase 2': $b < 3\cdot 5^y$. Use (VI) and the hypothesis of the induction (ii).

This completes the three cases. Now we can use the (old) recursion theorem to find a partial (but by its definition, primitive) recursive π such that $\pi(c) = 1$ if $c = 1$, $\pi(c) = 2^{\pi((c)_0)}$ if $c = 2^{(c)_0} \neq 1$, $\pi(c) = 3\cdot 5^d$ for d, s, p as above if $c = 3\cdot 5^{(c)_2}$, and $\pi(c) = 0$ otherwise.

Also, O' has indices for no more ordinals than O: *There is a primitive recursive function* $\varrho(c)$ *which maps* O' *homologously onto a subset of* O (cf. [6], 6.4). It suffices to take

$$\varrho(c) = \begin{cases} l \text{ if } c = 1, \\ 2^{\varrho((c)_0)} \text{ if } c = 2^{(c)_0} \neq 1, \\ 3\cdot 5^{S_1^2(g, r, c)} \text{ if } c = 3\cdot 5^{(c)_2}, \\ 0 \text{ otherwise,} \end{cases}$$

where r, g are Gödel numbers of ϱ, $\lambda rcn\, \Phi(r, \mathrm{pr}((c)_2, \langle n\rangle))$ (cf. § 10).

The results of this and the last section are good also for systems of ordinal notations relativized, e. g. to a one-place predicate Q or a two-place function $\varkappa$ (cf. §§ 3, 5, 7 and [7] § 30, [6] 6.4).

12. Numerous problems, some specific and some vague, arise in connection with the hierarchy we have described and similar hierarchies.

P 236. *Do the classes C_y as y ranges over O exhaust the general recursive functions? If not, can the subclass $\mathfrak{U}_{y\epsilon O} C_y$ of the general recursive functions be characterized in other terms?*

P 237. *Do various known classes of general recursive functions, e. g. Péter's k-recursive functions for a given $k > 1$, coincide with C_y, or perhaps instead with $\mathfrak{U}_{x<_O y} C_y$, for suitable y, s* (²)*?*

P 238. *For what ordinals $|y|$ do the classes C_y (or equivalently, the primitive recursive degrees $\mathbf{0}_{(y)}$) depend only on $|y|$* (²)*?*

P 239. *Can we say what should constitute inessential modifications in our method, uniform in y, of enumerating C_y, and then show that the hierarchy (i. e. the classes C_y, or equivalently the primitive recursive degrees $\mathbf{0}_{(y)}$) is invariant under such modifications?*

P 240. *If we start lower down (revising O to correspond), e. g. with the elementary functions, do the primitive recursive functions appear as* *some* C_y *or* $\mathfrak{U}_{x<_O y}\, C_y$? *More generally, how do the entire hierarchies compare for different initial classes with specified mode of generation (or in other terms, for different ways of generating a class from an assumed function, which is* $\lambda ba\ 0$ *for the lowest class)? In particular, how much smaller a class than the primitive recursive functions can one start with and get the same* $\mathfrak{U}_{y\epsilon O}\, C_y$?

REFERENCES

[1] W. Ackermann, *Zum Hilbertschen Aufbau der reellen Zahlen*, Mathematische Annalen 99 (1928), p. 118-133.

[2] — *Zur Widerspruchsfreiheit der Zahlentheorie*, ibidem 117 (1940), p. 162-194.

[3] P. Axt, *On a subrecursive hierarchy and primitive recursive degrees*, Ph. D. thesis, University of Wisconsin 1957.

[4] D. Hilbert, *Über das Unendliche*, Mathematische Annalen 95 (1926), p. 161-190.

[5] S. C. Kleene, *Introduction to metamathematics*, New York and Toronto, Amsterdam and Groningen 1952.

[6] — *Arithmetical predicates and function quantifiers*, Transactions of the American Mathematical Society 79 (1955), p. 312-340.

[7] — *On the forms of the predicates in the theory of constructive ordinals (second paper)*, American Journal of Mathematics 77 (1955), p. 405-428.

[8] — *Hierarchies of number-theoretic predicates*, Bulletin of the American Mathematical Society 61 (1955), p. 193-213.

[9] — and E. L. Post, *The upper semi-lattice of degrees of recursive unsolvability*, Annals of Mathematics 59 (1954), p. 379-407.

[10] J. Myhill, *A stumblingblock in constructive mathematics*, abstract, Journal of Symbolic Logic 18 (1953), p. 190-191.

[11] R. Péter, *Konstruktion nichtrekursiver Funktionen*, Mathematische Annalen 111 (1935), p. 42-60.

[12] — *Rekursive Funktionen*, Budapest 1951.

[13] — *Probleme der Hilbertschen Theorie der höheren Stufen von rekursiven Funktionen*, Acta Mathematica Academiae Scientiarum Hungaricae 2 (1951), p. 247-274.

[14] — *Rekursive Definitionen, wobei frühere Funktionswerte von variabler Anzahl verwendet werden*, Publicationes Mathematicae 3 (1953), p. 33-70.

[15] N. A. Routledge, *Ordinal recursion*, Proceedings of the Cambridge Philosophical Society 49 (1953), p. 175-182.

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LOCAL ORIENTABILITY

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It is the purpose of this paper* 1° to clarify and extend the notion of *local orientability* which was defined on p. 281-282 of my book [7], and 2° to apply the results obtained to establish a definition of orientability for an n-dimensional generalized manifold ($= n$-gm) which is the exact analogue of the Poincaré definition (1).

It is hoped that these results will contribute to the solutions of a number of unsolved problems concerning manifolds (see, for example, [7], p. 382-383, problems 4.1, 4.5).

1. Some basic lemmas. For the proofs given below it is necessary to have the following definitions and lemmas, which are inserted at this point for convenience of reference.

1.1. LEMMA. *In an n-dimensional space S, if P is an open set with compact closure, and γ^n is a cycle mod $S-P$, then there exists a minimal closed (rel. P) subset F of P such that γ^n is carried by $F \cup (S-P)$.*

Proof. The portion of γ^n on $\overline{P}$ is a cycle Z^n mod $F(P)$ on $\overline{P}$. As $\overline{P}$ is compact, there exists by [7], p. 205-6, Lemma 2.3, a minimal closed subset F' of $\overline{P}$ that contains $F(P)$ such that $Z^n \sim 0 \bmod F'$; and by [7], p. 206, Lemma 2.6, F' is unique and a closed carrier of Z^n. Let $F = F' \cap P$. Since $\gamma^n \sim Z^n \bmod S-P$, the lemma follows.

1.2. LEMMA. *If S is an n-dimensional locally compact space, then every infinite cycle Γ^n of S has a unique minimal closed carrier.*

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Terminology and notation are that of my book [7].

(1) This definition states that an n-manifold, without boundary, whose elements are oriented k-cells ($k = 0, 1, \ldots, n$) is orientable if every "closed chain" of cells $\sigma_i^n, \sigma_{j(1)}^{n-1}, \sigma_{k(1)}^n, \ldots, \sigma_{k(m)}^n, \sigma_{j(m+1)}^{n-1}, \sigma_{k(m+1)}^n, \ldots, \pm\sigma_i^n$ in which $\sigma_{k(m)}^n$ and $\sigma_{k(m+1)}^n$ are oppositely related to $\sigma_{j(m+1)}^{n-1}$ had $+\sigma_i^n$ as „end" element. See [6], § 8.