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ON A COMBINATORICAL PROBLEM

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1. This paper deals with the following problem:

A matrix is said *to be of type* $(R, K)_s$, where $0 \leqslant s \leqslant RK$, if it has R rows and K columns and if s of its elements are 1 and the rest 0. Now let R, K, r, k, where $1 \leqslant r \leqslant R$ and $1 \leqslant k \leqslant K$, be four given natural numbers. Which is the greatest number $s = A(R, K, r, k)$, such that there exists a matrix of the type $(R, K)_s$, which does not contain any minor of the type $(r, k)_{rk}$, *i. e.* a minor with r rows and k columns and all elements equal to 1?

This problem was (for $R = K$, $r = k$) raised by K. Zarankiewicz in [3]. It is properly a logical problem and can be formulated in the following way: Let E and F be two sets with R and K elements, respectively. How many elements s can a relation between E and F (*i. e.* a subset G of $E \times F$) contain, without containing any subset of the type $E' \times F'$, where E' and F' are subsets of E and F with r and k elements, respectively? In the following, however, we use the matrix formulation.

In [2] T. Kővari, V. T. Sós and P. Turán proved that

$$A(n, n, j, j) < jn + [(j-1)^{1/j} n^{(2j-1)/j}], \tag{1}$$

where $[x]$ denotes the integral part of x. They also showed the asymptotic formula

$$\lim_{n\to\infty} A(n, n, 2, 2) n^{-3/2} = 1. \tag{2}$$

The same method as was used in [2] to prove (1) can also, as mentioned there, be used to give an estimate of $A(R, K, r, k)$. This gives

$$A(R, K, r, k) \leqslant (r-1)K + (k-1)^{1/r} K^{1-1/r} R \tag{3}$$

after a slight sharpening of the estimates influencing the first term in the second member.

In this paper I will in section 2 give a special method for estimating $A(R, K, 2, k)$ from above, which gives another estimate than that obtained by specializing (3). Section 3 contains an asymptotic formula for $A(n, n, 2, 3)$ and section 4 an estimate of $A(n, n, 2, 2h)$ from below. In section 5 some asymptotic formulas for rectangular matrices are discussed.

Relations (3) and (8), as well as relations (4) and (8), imply (2).

2. THEOREM 1. *For $k \geqslant 1$ one has*

$$A(R, K, 2, k) \leqslant \tfrac{1}{2}K + \big((k-1)KR(R-1) + K^2/4\big)^{1/2}. \tag{4}$$

Proof. We denote the row-vectors of a matrix M of the type $(R, K)_s$ by B_ν, $\nu = 1, 2, \ldots, R$. If $V = \sum_{\nu=1}^{R} B_\nu$ we have

$$V^2 = \sum_{\nu=1}^{R} B_\nu^2 + \sum_{\nu \neq \mu} B_\nu B_\mu = s + \sum_{\nu \neq \mu} B_\nu B_\mu, \tag{5}$$

since B_ν^2 is equal to the number of ones in the ν-th row. Further if we put $V = (a_1, a_2, \ldots, a_K)$, we have $\sum_{\nu=1}^{K} a_\nu = s$. Hence the Cauchy inequality gives $s^2 \leqslant KV^2$ and from (5) it therefore follows that

$$s^2/K \leqslant s + \sum_{\nu \neq \mu} B_\nu B_\mu.$$

Now if the matrix M does not contain any minor of the type $(2, k)_{2k}$, it is immediately clear that $B_\nu B_\mu \leqslant k-1$ for $\nu \neq \mu$, and hence $s^2/K \leqslant \leqslant s + (k-1)R(R-1)$, which implies (4).

3. THEOREM 2. *One has*

$$\lim_{n\to\infty} A(n, n, 2, 3) n^{-3/2} = \sqrt{2}. \tag{6}$$

Proof. From (3) it follows that $A(n, n, 2, 3) \leqslant n + \sqrt{2}\, n^{3/2}$.

It is now possible to get a lower estimate by a modification of the method used in [2]. Let p be a prime $\geqslant 5$ and let the numbers N, a, b vary in the following way:

$$N = 1, 2, \ldots, (p-3)/2, \quad a = 1, 2, \ldots, p, \quad b = 1, 2, \ldots, (p-1)/2. \tag{7}$$

Let us further define $\langle x \rangle_p$ as the remainder resulting from the division of x by p, so that $\langle x \rangle_p \equiv x(p)$ and $0 \leqslant \langle x \rangle_p < p$.

We shall now construct a square matrix M_3 of order $p(p-1)/2$. We enumerate the columns by the numbers $1, 2, \ldots, p(p-1)/2$ and denote the rows by the *pairs* of numbers (a, b), where a and b vary as in (7).

We then prescribe that there shall be ones in the row (a, b) in all places with the column-numbers

$$c_N = Np + \langle aN + b \rangle_p$$

and with the column-numbers

$$d_N = Np + \langle aN - b \rangle_p,$$

where $N = 1, 2, \ldots, (p-3)/2$. (Note that $1 < c_N < p(p-1)/2$ and $1 < d_N < p(p-1)/2$.) Obviously $c_N \neq c_L$ if $N \neq L$. Further, $c_N \neq d_L$ for all N and L. For $c_N = d_L$ would first imply that $N = L$. Then we should have $aN + b \equiv aN - b(p)$ and thus $2b \equiv 0(p)$, which is impossible. To sum up, this means that there are

$$p\,\frac{p-1}{2}\,2\,\frac{p-3}{2} = \frac{p^3}{2} + O(p^2)$$

ones in the matrix M_3.

Can M_3 contain any minor of the type $(2,3)_6$?

Let us regard the two rows denoted by (a, b) and (a', b'), where (a, b) and (a', b') are two non-identical pairs of numbers. If there are ones with the same column-numbers in these two rows, they must correspond to the same value of N. Therefore we have to count the total number of solutions in N of the four congruences

(I) $aN_1 + b \equiv a'N_1 + b'(p)$, (III) $aN_3 + b \equiv a'N_3 - b'(p)$,

(II) $aN_2 - b \equiv a'N_2 - b'(p)$, (IV) $aN_4 - b \equiv a'N_4 + b'(p)$.

Each one of these congruences has at most one solution. If $a \neq a'$, this is immediately clear. If $a = a'$, the existence of a solution in the first or second case would imply $b = b'$ contrary to the assumption. In the third or fourth case it would imply $b + b' \equiv 0(p)$, which is also impossible. Hence there is *no* solution if $a = a'$.

After this remark we observe that the first two congruences cannot both be solvable, for on adding them we should then get

$$a(N_1 + N_2) \equiv a'(N_1 + N_2)(p),$$

where we can assume $a \neq a'$. But then $N_1 + N_2 \equiv 0(p)$, which is impossible since N varies as in (7). The same argument holds for the last two congruences. Therefore the total number of solutions in N is $\leqslant 2$, and the matrix M_3 does not contain any minor of the type $(2,3)_6$. This proves that if p is prime $\geqslant 5$ then

$$A\left(\frac{p(p-1)}{2}, \frac{p(p-1)}{2}, 2, 3\right) \geqslant \frac{p(p-1)(p-3)}{2}.$$

Now $A(n, n, 2, 3)$ increases with n and as $p_\nu/p_{\nu+1} \to 1$ when $\nu \to \infty$, where p_ν is the ν-th prime, we get

$$\varliminf_{n\to\infty} A(n, n, 2, 3) n^{-3/2} \geqslant \sqrt{2},$$

and hence the proof is finished.

4. THEOREM 3. *For integral $h \geqslant 1$ one has*

$$\varliminf_{n\to\infty} A(n, n, 2, 2h) n^{-3/2} \geqslant h^{1/2}. \tag{8}$$

Proof. For $h = 1$ and $h = 2$ the estimation is contained in (2) and (6), respectively. For in the last case we have

$$\varliminf_{n\to\infty} A(n, n, 2, 4) \geqslant \lim_{n\to\infty} A(n, n, 2, 3) = \sqrt{2}.$$

To prove it generally we choose p prime $>3h$ and let the numbers N, a, b vary in the following way:

$$N = 1, 2, \ldots, p-3h, \qquad a = 1, 2, \ldots, p, \qquad b = 1, 2, \ldots, [p/h].$$

We now construct a square matrix of order $p[p/h]$. As before the columns are enumerated by the integers $1, 2, \ldots, p[p/h]$ and the rows by the pairs (a, b). We prescribe that in the row (a, b) there shall be ones in the places with the column-numbers

$$e_N = N\left(\left[\frac{p}{h}\right]+1\right) + \left[\frac{\langle aN+bh\rangle_p}{h}\right]$$

and zeros elsewhere. (If here we put $h = 1$, we get essentially the construction used in [2] to prove (2). Observe that $e_N \neq e_L$ if $N \neq L$ and that $1 < e_N < p[p/h]$.)

It is now possible to prove that this matrix cannot contain any minor of the type $(2, 2h)_{4h}$. Suppose that the contrary were true and that the two rows were (a, b) and (a', b'). The ones with the same column-number must correspond to the same value of N and therefore

$$\left[\frac{\langle aN+bh\rangle_p}{h}\right] = \left[\frac{\langle a'N+b'h\rangle_p}{h}\right]$$

would be true for $2h$ different values of N. But if the common value of the two members above is denoted by H_N, we get

$$aN+bh \equiv H_N h + s_N(p), \qquad a'N+b'h \equiv H_N h + s'_N(p), \tag{9}$$

where $0 \leqslant s_N \leqslant h-1$ and $0 \leqslant s'_N \leqslant h-1$. Subtracting the congruences we get

$$(a-a')N + (b-b')h \equiv s_N - s'_N \equiv f_N(p), \tag{10}$$

where $-(h-1) \leqslant f_N \leqslant (h-1)$.

First we observe that $N = N_1$ and $N = N_2 \neq N_1$ cannot correspond to the same value of f_N. For then $(a-a')(N_1 - N_2) \equiv 0(p)$, and this implies $a = a'$, so that (10) gives $(b-b')h - f_N \equiv 0(p)$.

Now $|(b-b')h - f_N| \leqslant (p/h-1)h + h - 1 = p-1$, so that $(b-b')h - f_N = 0$. But this means that $f_N = 0$ and $b = b'$, which is impossible, since we also had $a = a'$.

As there are only $2h-1$ possible values of f_N and each one of them corresponds to one N at most the system (9) cannot have $2h$ solutions for N.

We have now constructed a square matrix of order $p[p/h]$. Since there are $p-3h$ elements equal to 1 in each row, the matrix contains $p^3/h + O(p^2)$ elements equal to 1. Since the matrix does not contain any minor of the type $(2, 2h)_{4h}$, we deduce Theorem 3 in the same way as Theorem 2.

Finally, since $A(n, n, 2, k)$ increases with k it follows immediately from (8) and (3) that

$$[k/2]^{1/2} \leqslant \varliminf_{n\to\infty} A(n, n, 2, k) n^{-3/2} \leqslant \varlimsup_{n\to\infty} A(n, n, 2, k) n^{-3/2} \leqslant (k-1)^{1/2} \tag{11}$$

for all integers $k \geqslant 1$.

5. We finish with some remarks concerning rectangular matrices. For fixed R it follows from (3) that

$$\varlimsup_{K\to\infty} A(R, K, r, k) K^{-1} \leqslant r-1.$$

But the matrix with all its elements in $r-1$ rows equal to 1 and the rest equal to 0 shows that $A(R, K, r, k) \geqslant (r-1)K$, so that

$$\lim_{K\to\infty} A(R, K, r, k) K^{-1} = r-1. \tag{12}$$

This follows also from the exact formula

$$A(R,K,r,k) = (r-1)K+(k-1)\binom{R}{r} \quad \text{for all} \quad K \geqslant (k-1)\binom{R}{r},$$

proved by K. Čulík in [1].

Let us now for $r = 2$ consider the case when $K \to \infty$ and R/K converges to a fixed number t. Then (3) shows that

$$\overline{\lim_{\substack{K\to\infty \\ R/K\to t}}} A(R,K,2,k)K^{-3/2} \leqslant t(k-1)^{1/2} \quad \text{for all} \quad t \geqslant 0.$$

On the other hand, we observe that in the matrices M_2, used in the proof of (2) (see [2]) and M_3, used in the proof of Theorem 2, all rows contained the same number of ones. This shows that

$$g_k(t) = \lim_{\substack{K\to\infty \\ R/K\to t}} A(R,K,2,k)K^{-3/2} = t(k-1)^{1/2}$$

for $k = 2$ or 3 and $0 \leqslant t \leqslant 1$.

The function g_2 can easily be determined for all t. For if $t > 1$ one has

$$\lim_{\substack{K\to\infty \\ R/K\to t}} A(R,K,2,2)K^{-3/2} = \lim_{\substack{R\to\infty \\ K/R\to 1/t}} A(K,R,2,2)R^{-3/2}t^{3/2},$$

so that $g_2(t) = t^{3/2}g_2(1/t)$. Hence

$$g_2(t) = \begin{cases} t & \text{for} \quad 0 \leqslant t \leqslant 1, \\ t^{1/2} & \text{for} \quad t > 1. \end{cases}$$

As for general k we first put $k-1$ matrices M_2 beside each other and conclude that from (2) and (3) follows

(13) $$g_k(t) = t(k-1)^{1/2} \quad \text{for all} \quad k \geqslant 2 \text{ and } 0 \leqslant t \leqslant 1/(k-1).$$

Now $R_2^{-1}A(R_2,K,2,k) \leqslant R_1^{-1}A(R_1,K,2,k)$ if $R_2 \geqslant R_1$ and this inequality shows that the function $h_k(t) = t^{-1}g_k(t)$ is non-increasing where it exists. For all $k \geqslant 2$ the formula (13) shows that h_k is constant in a certain interval $0 \leqslant t \leqslant a_k$ and one could therefore ask for the greatest a_k with this property. Some investigations in this direction follow below.

If $k \geqslant 3$ is odd we put $(k-1)/2$ of the matrices M_3 beside each other and conclude that (13) is valid in the extended interval $0 \leqslant t \leqslant 2/(k-1)$. Finally, if $k-1$ is a square, we will prove that (13) is valid in $0 \leqslant t \leqslant$ $\leqslant 1/(k-1)^{1/2}$.

THEOREM 4. *For integral $s \geqslant 1$ and $0 \leqslant t \leqslant 1/s$ one has*

(14) $$\lim_{\substack{K\to\infty \\ R/K\to t}} A(R,K,2,s^2+1)K^{-3/2} = ts.$$

Proof. To get an estimate from below one can proceed as follows: Let p be a prime $> s$ and let the numbers N, L, a, b vary in this way:

(15) $$\begin{aligned} N &= 1,2,\ldots,p-1, & a &= 1,2,\ldots,p, \\ L &= 1,2,\ldots,s, & b &= 1,2,\ldots,[p/s]. \end{aligned}$$

Now define a matrix with p^2 columns and $p[p/s]$ rows as follows: Enumerate the columns by the integers $1,2,\ldots,p^2$ and the rows by the pairs (a,b). Further prescribe that in the row (a,b) there shall be ones in all places with the column-numbers

$$Np+\langle aN+bs+L\rangle_p,$$

where N and L vary as in (15). This gives $(p-1)s$ elements equal to 1 in every row and hence $p^3+O(p^2)$ in the whole matrix.

Consider now special choices of L, say L in the row (a,b) and L' in the row (a',b'). There are s^2 possibilities to choose the pair L, L' and it is easy to see that to each of these choices corresponds at most one solution in N of the congruence

$$aN+bs+L \equiv a'N+b's+L'.$$

Hence the matrix constructed does not contain any minor of the type $(2, s^2+1)_{2(s^2+1)}$. The rest of the proof runs as above.

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