ON THE DECOMPOSITION OF A SEGMENT INTO CONGRUENT SETS AND RELATED PROBLEMS

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In this paper we deal with decompositions of some geometrical figures into disjoint congruent sets. The origin of these problems is Vitali's well known decomposition of the circle into $\aleph_0$ congruent sets.

It has been proved by J. von Neumann [4] that each of the intervals

$$(0,1) = \{x\colon 0 < x < 1\},\ \langle 0,1) = \{x\colon 0 \leqslant x < 1\},\ \langle 0,1\rangle = \{x\colon 0 \leqslant x \leqslant 1\}$$

can be decomposed into $\aleph_0$ disjoint congruent sets. This theorem is generalized here in the following way:

THEOREM 1[1]). *If $\aleph_0 \leqslant \mathfrak{m} \leqslant 2^{\aleph_0}$ then each of the intervals $(0,1)$, $\langle 0,1)$, $\langle 0,1\rangle$ is the sum of $\mathfrak{m}$ disjoint sets, congruent to each other by translation, each of the power $2^{\aleph_0}$* [2]).

The idea of the proof which follows is the same as in [4], but the construction is simplified.

LEMMA 1. *If $\aleph_0 \leqslant \mathfrak{m} \leqslant 2^{\aleph_0}$, then there exists a set F satisfying the conditions*

(1) $F \subset \langle a, \beta\rangle$, $a \in F$, $\beta \in F$, *where* $0 < a < \beta < 1$;

(2) *If* $a \leqslant x < y \leqslant \beta$ *then* $\overline{\overline{\langle x, y\rangle \cap F}} = \mathfrak{m}$;

(3) *The elements of F are irrational numbers*;

(4) $F \subset H$ *where H is a Hamel basis and* $\overline{\overline{H \setminus F}} = 2^{\aleph_0}$.

The proof is an easy construction by means of the axiom of choice.

LEMMA 2. *If F satisfies (1), (2), (3) and (4) then each of the intervals $(0,1)$, $\langle 0,1)$, $\langle 0,1\rangle$ is a sum of $2^{\aleph_0}$ disjoint sets congruent to F by translation.*

Proof. For any linear set X and any real number τ we put

(5) $$X_\tau = \{x+\tau\colon x \in X\}.$$

[1]) This result was announced in [2], p. 350, footnote [8]).
[2]) Concerning the case $\mathfrak{m} = 2^{\aleph_0}$ cf. [6].

Let I be the considered interval ($(0,1)$ or $\langle 0,1)$ or $\langle 0,1\rangle$). Let F^*_τ be the smallest additive group containing F_τ. We put $I^*_\tau = I \cap F^*_\tau$. Then $\bigcup_\tau I^*_\tau = I$, and $I^*_{\tau_1} \cap I^*_{\tau_2}$ is empty or $I^*_{\tau_1} = I^*_{\tau_2}$.

By (2) $\overline{\overline{I^*_\tau}} = \mathfrak{m}$ for every τ.

By (4) for different $\tau \in H \setminus F$ the sets I^*_τ are disjoint. Then for proving the lemma it is enough to verify that every set I^*_τ can be decomposed into disjoint sets congruent to F by translation.

Let us well-order I^*_τ into a sequence $\{p_\xi\}_{\xi<\mathfrak{m}}$, and suppose that $p_0 = 0$ if $0 \in I^*_\tau$ and $p_0 = 1$ if $1 \in I^*_\tau$ (by (3) these possibilities are exclusive).

Suppose that for an ordinal number η ($\bar{\eta} < \mathfrak{m}$) there exists such a set T that

(6) for different $t \in T$ the sets F_t are disjoint,

(7) $\{p_\xi\}_{\xi<\eta} \subset \bigcup_{t\in T} F_t \subset I^*_\tau$.

We shall prove the existence of a set T' with the properties (6) and (7) but such that

$$p_\eta \in \bigcup_{t\in T'} F_t.$$

This of course will be an inductive proof of the existence of the required decomposition of I^*_τ.

Clearly, we can consider only the case when

(8) $$\overline{\overline{T}} \leqslant \bar{\eta} < \mathfrak{m} \quad \text{and} \quad p_\eta \,\bar{\in}\, \bigcup_{t\in T} F_t$$

and prove the existence of such a t' that $T' = T \cup \{t'\}$ has the required properties.

If $\eta = 0$ and $p_0 = 0$ or 1 then $t' = -a$ or $1-\beta$ respectively. Then (6) is trivial and (7) follows from (1).

If $\eta > 0$ or $0 \neq p_0 \neq 1$ then there exists a set M such that

(9) $$\overline{\overline{M}} = \mathfrak{m} \text{ and, for every } t' \in M,\ p_\eta \in F_{t'} \subset I^*_\tau.$$

In fact, by (1) and (2) there exist $\mathfrak{m}$ numbers t' such that $p_\eta \in F_{t'} \subset I$; but $p_\eta \in F_{t'}$ implies also $F_{t'} \subset F^*_\tau$.

Now we shall prove that

(10) *For every $t \in T$ the set of those $t' \in M$ for which $F_t \cap F_{t'}$ is non empty has at most one element.*

For those t' we have $p_\eta = f_1 + t'$ for some $f_1 \in F$ (by (9)) and $f_2 + t = f_3 + t'$ for some $f_2, f_3 \in F$. Then $t' = p_\eta - f_1 = t - f_3 + f_2$ for some $f_1, f_2, f_3 \in F$.

By (4) F is a set of free generators for the group F^*. Now p_η and t are fixed and consequently f_2 is fixed. By (8) $p_\eta \neq t + f_2 \in F_t$ and consequently f_1 and f_3 are also fixed. Hence if such a t' exists, it is unique.

From (8), (9) and (10) we infer the existence of such a $t' \epsilon M$ that

$$F_{t'} \cap \bigcup_{t \epsilon T} F_t$$

is empty, q. e. d.

Proof of theorem 1. Using lemmas 1 and 2 we obtain the existence of such sets F and E that (cf. (4))

$$I = \bigcup_{u \epsilon E} F_u = \bigcup_{u \epsilon E} \bigcup_{v \epsilon F} \{u+v\} = \bigcup_{v \epsilon F} E_v,$$

where $\overline{\overline{E}} = \overline{\overline{E_v}} = 2^{\aleph_0}$, $\overline{\overline{F}} = \mathfrak{m}$, and F_u and consequently E_u are disjoint, q. e. d.

It is clear that

$$\langle 0,1) = \bigcup_{i=0}^{n-1} \left\langle \frac{i}{n}, \frac{i+1}{n} \right) \quad \text{for} \quad n = 1, 2, \ldots$$

but for no natural number $n > 1$ any of the intervals $(0,1)$, $\langle 0,1 \rangle$ can be decomposed into n disjoint sets congruent by translation (see Gustin [1]).

The following problem of H. Steinhaus remains unsolved:

P 193. Is it true that for each integer $n > 1$ none of the intervals $(0,1)$, $\langle 0,1 \rangle$ can be divided into n disjoint sets congruent by translation or by rotation?

W. Sierpiński ([6], p. 63) has pointed out a positive answer for $n = 2$. A. Schinzel has given a proof for $n = 3$.

It is interesting to consider the same question for other geometrical figures. Clearly, by theorem 1, statements analogous to this theorem hold for the straight line and the circle and for Cartesian products in which such spaces occur as factors. I have proved [1] the existence of such a set E on the sphere $S_2 (x^2+y^2+z^2 = 1)$, that for every cardinal number $\mathfrak{m}$ for which $2 < \mathfrak{m} \leqslant 2^{\aleph_0}$ S_2 can be divided into $\mathfrak{m}$ disjoint sets congruent to E by rotation (note that this implies the same result for the space E^3 without one point. The same holds for the space E^3 [3]). The same holds also for all the spheres S_n $(n \geqslant 2)$ [4]) except S_4 for which the problem remains open). Of course this set E is not measurable and it is constructed by means of the axiom of choice.

It is not known if we can prove without using the axiom of choice that the sphere S_2 can be decomposed into 3 disjoint congruent sets[5]).

[3]) By theorem 2 of my paper [2].

[4]) This follows from some unpublished results of T. Dekker. See also [2].

[5]) Cf. P 166, Coll. Math. 4 (1957), p. 240.

An analogous problem for solid spheres open or closed of dimensions 2, 3, ... has not been studied. (Note that a negative result concerning paradoxical decompositions of the solid open and closed spheres of dimension 3 was given by R. M. Robinson [3], § 6.)

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