Dann ist $S(x) = x(x+1)/2 - x$. Wird $f(n) = 2\sqrt{n}$ angenommen, so wird $n \leqslant S(f(n))$ und es gilt also $A(n) = (\beta-\alpha)n + O(\sqrt{n})$.

2. $a_n = a^n$ ($a > 1$, ganz).

Jetzt ist $S(x) = (a^{x+1}-1)/(a-1) - x$. Setzt man

$$f(n) = \frac{\log\left(\frac{1}{2}(a-1)n+1\right)}{\log a} - 1,$$

so gilt

$$S(f(n)) = \tfrac{1}{2}n + O(\log n),$$

$$a_{[f(n)]} = a^{[f(n)]} > a^{f(n)-1} = \frac{\frac{1}{2}(a-1)n+1}{a^2} \neq o(n).$$

Die Folge (1) ist deshalb für $a_n = a^n$ nicht gleichverteilt.

3. $a_n = p_n$ ($p_1, p_2, \ldots$ die Primzahlenfolge).

Dann ist

$$S(n) = \sum_{\nu=1}^{n} p_\nu - n = \sum_{p \leqslant p_n} p - n,$$

daher $S(x) \sim \frac{1}{2}p_x^2/\log p_x$ [3]) und, wegen $p_x \sim x\log x$,

$$S(x) \sim \tfrac{1}{2}x^2 \log x.$$

Setzt man $f(n) = 3\sqrt{n/\log n}$ und ist n hinreichend groß, so erhält man $S(f(n)) \leqslant n$ und daraus

$$A(n) = (\beta-\alpha)n + O(\sqrt{n\log n}).$$

Wird $f(n) = \sqrt{n/\log n}$ gesetzt, so hat man $S(f(n)) \leqslant n$ ($n \geqslant n_0$) und daraus $A(n) = (\beta-\alpha)n + \Omega(\sqrt{n\log n})$ für jedes Intervall $[\alpha, \beta) \neq [0,1)$.

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[3]) Siehe E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Leipzig-Berlin 1909, Bd I, S. 226.

ON THE EQUATION $x^3+y^3 = 2z^3$

BY

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The purpose of this paper is a solution of the following problem[1]) set up by prof. W. Sierpiński: Does the infinite sequence $1^3, 2^3, 3^3, \ldots$ contain an arithmetic progression with three (or more) terms? In other words, does there exist a solution of the equation $x^3+y^3 = 2z^3$ with natural x, y and z?

By the way I want to give elementary proofs of the following theorems:

THEOREM 1. *An equation*

$$27d^4 + 9a^2d^2 + a^4 = l^2 \tag{1}$$

has no solutions in natural numbers if $(a, 3) = 1$.

THEOREM 2. *An equation*

$$x^3+y^3 = 2z^3 \tag{2}$$

has no solutions in integers if $x \neq y$, *and* $z \neq 0$.

From Theorem 2 the following inferences are deduced:

i. *There are no three natural cubics in arithmetical progression.*

ii. *The equations* $x^3 - 2z^3 = \pm 1$ *have no integer solutions, except* $x = \pm 1$, $z = 0$, *and* $x = z = \pm 1$.

In fact, according to Theorem 2 the solutions of (2) exist only in case of: $z = 0$, or $x = y$. If $z = 0$, then on account of $y = \pm 1$ we have $x = \pm 1$; if, however, $x = y$ then $x = \pm 1$ and $z = \pm 1$.

iii. *A triangular number* > 1 *is never a natural cube.*

Really, if $m(m+1)/2 = n^3$, then $m(m+1) = 2n^3$ and $m = 2k$ or $m+1 = 2k$. If $m = 2k$, then $k(2k+1) = n^3$, whence $k = n_1^3$, $2k+1 = n_2^3$ and $n_2^3 - 2n_1^3 = 1$, which is impossible for natural n_1, n_2. If, however, $m+1 = 2k$, then $k(2k-1) = n^3$, and thus $k = n_1^3$, $2k-1 = n_2^3$ and

[1]) W. Sierpiński, *Remarques sur les progressions arithmétiques*, Coll. Math. 3 (1954), p. 44-49; P 116, p. 45.

$n_2^3-2n_1^3 = -1$, which is possible for natural numbers only when $n_1 = n_2 = 1$, whence $k = 1$, $m = 1$.

iv. *The only natural solution of the equation* $x^2-y^3 = 1$ *is* $x = 3$, $y = 2$.

In fact, we have $(x-1)(x+1) = y^3$, so for $x = 2k$ it would be $2k-1 = y_1^3$, $2k+1 = y_2^3$ and $y_2^3-y_1^3 = 2$, whence $y_2^2+y_2y_1+y_1^2 \mid 2$, which is impossible for natural y_1, y_2.

If, however, $x = 2k-1$ with natural k, then $y = 2y_1$, $(k-1)k = 2y_1^3$ and the alternative: $k = y_2^3$, $k-1 = 2y_3^3$, or $k = 2y_3^3$, $k-1 = y_2^3$, whence $y_2^3-2y_3^3 = \pm 1$. The only natural solution is $y_2 = y_3 = 1$, and thus $k = 2$, $x = 3$, $y = 2$.

I. First we shall show the relation existing between equations (1) and (2).

Putting aside the obvious cases, we shall look for integer solutions of equations (2) such as $x \neq y$, $xyz \neq 0$.

We can accept as well $(x, y) = 1$ and x, y odd, for in the case $x = 2x_1$ we have $y = 2y_1$ and $z = 2z_1$ and the equation $x_1^3+y_1^3 = 2z_1^3$, whence, after a certain number of such substitutions, the equation $x^3+y^3 = 2z^3$, where x, y are odd: in the case of $(x, y) = d > 1$ and $(d, 2) = 1$ we have $x = dx_1$, $y = dy_1$, $z = dz_1$, whence $x_1^3+y_1^3 = 2z_1^3$, where $(x_1, y_1) = 1$.

Now we substitute $(x+y)/2 = u$, $(x-y)/2 = v$ in (2). We get $(u, v) = 1$ and

(3) $$u(u^2+3v^2) = z^3 \quad \text{where} \quad uvz \neq 0.$$

1. If $(u, 3) = 1$ so $u = z_1^3$, $u^2+3v^2 = z_2^3$ and $(z_1, z_2) = 1$. We have then $z_1^6+3v^2 = z_2^3$, $z_2^3-z_1^6 = 3v^2$ and

$$(z_2-z_1^2)[(z_2-z_1^2)^2+3z_2z_1^2] = 3v^2.$$

Let us set $z_2 = k+z_1^2$, then $(k, z_1) = 1$ and $k(k^2+3kz_1^2+3z_1^4) = 3v^2$. Hence $k = 3k_1$, $3k_1(9k_1^2+9k_1z_1^2+3z_1^4) = 3v^2$, and thus $v = 3v_1$, $k_1 = 3k_2$ and

(4) $$k_2(27k_2^2+9k_2z_1^2+z_1^4) = v_1^2 \quad \text{where} \quad (k_2, z_1) = 1.$$

Since $(k_2, 27k_2^2+9k_2z_1^2+z_1^4) = 1$, we obtain $k_2 = k_3^2$ and

(5) $$27k_3^4+9k_3^2z_1^2+z_1^4 = l_1^2 \quad \text{with} \quad (z_1, 3) = 1.$$

2. If $3|u$ then $(v, 3) = 1$ and $u = 3u_1$, $z = 3z_1$, $27u_1^3+9u_1v^2 = 27z_1^3$. Hence $u_1 = 3u_2$ and $27u_2^3+u_2v^2 = z_1^3$ or

(6) $$u_2(27u_2^2+v^2) = z_1^3.$$

Since $(u_2, v) = 1$ then $u_2 = a^3$, $27u_2^2+v^2 = b^3$ and $(a, b) = 1$, $(b, 3) = 1$ and

(7) $$27a^6+v^2 = b^3.$$

By setting $b-3a^2 = k$, we obtain $(k, 3) = 1$ and

(8) $$v^2 = k(k^2+9a^2k+27a^4).$$

Hence $k = k_1^2$ and $k^2+9a^2k+27a^4 = l^2$ so that

(9) $$k_1^4+9a^2k_1^2+27a^4 = l_1^2 \quad \text{with} \quad (k_1, 3) = 1.$$

Both in the case 1 and in 2 there exist integer solutions of the equation $x^3+y^3 = 2z^3$ where $x \neq y$, $xyz \neq 0$ only if there exist natural solutions of the equation of the form

$$27d^4+9a^2d^2+a^4 = l^2 \quad \text{where} \quad (a, 3) = 1.$$

II. Solution of equation (1) if $(a, 3) = 1$. We suppose a lexicographically ordered set of natural solutions (l, a, d) of this equation and seek the smallest one. Then we can assume $(a, d) = 1$, whence also $(a, l) = (d, l) = 1$.

Equation (1) leads to

$$27\left(\frac{d^2}{l}\right)^2+9\frac{a^2}{l}\frac{d^2}{l}+\left(\frac{a^2}{l}\right)^2 = 1$$

or to an equation for rational x, y

(11) $$27x^2+9xy+y^2 = 1.$$

Taking $y = mx-1$ we obtain all rational solutions of (11) as

$$x = \frac{2m+9}{m^2+9m+27}, \quad y = \frac{m^2-27}{m^2+9m+27},$$

where m is an arbitrary rational.

Evidently we may exclude $m = 0$ because $y > 0$. Thus if $m = p/q$, then $pq \neq 0$, $(p, q) = 1$ and

(12) $$\frac{d^2}{l} = \frac{2pq+9q^2}{p^2+9pq+27q^2}, \quad \frac{a^2}{l} = \frac{p^2-27q^2}{p^2+9pq+27q^2}.$$

On account of $(d^2, l) = (a^2, l) = 1$ we have

$$(2pq+9q^2, p^2+9pq+27q^2) = (p^2-27q^2, p^2+9pq+27q^2) = \delta.$$

By setting $p = 3^sp_1$ where $(p_1, 3) = 1$, we obtain

(13) $$(3^s\cdot 2p_1q+9q^2, 3^{2s}p_1^2+9\cdot 3^sp_1q+27q^2) = (3^{2s}p_1^2-27q^2, 3^{2s}p_1^2+9\cdot 3^sp_1q+27q^2)$$

with $(q, 3) = 1$.

For $s = 1$ formula (13) is impossible mod 9.

For $s = 2$ or $p = 9p_1$ we have

$$(p^2-27q^2,\ p^2+9pq+27q^2) = 27(3p_1^2-q^2,\ 3p_1^2+3p_1q+q^2).$$

But

$$(3p_1^2-q^2,\ 3p_1^2+3p_1q+q^2) = (6p_1^2+3p_1q,\ 3p_1^2+3p_1q+q^2)$$
$$= (2p_1+q,\ 3p_1^2+3p_1q+q^2) = (2p_1+q,\ 3p_1^2+p_1q) = (2p_1+q,\ 3p_1+q) = 1.$$

Thus $l = 3p_1^2+3p_1q+q^2$ and $a = 3p_1^2-q^2$, which is impossible mod 4.

If $s \geqslant 3$ then (13) is impossible mod 27.

It follows that $s = 0$, $(p, 3) = 1$ and on account of $(p, q) = 1$

$$(2pq+9q^2,\ p^2+9pq+27q^2) = (p^2+3pq,\ 2pq+9q^2)$$
$$= (p+3q,\ 2p+9q) = (p, 3) = 1,$$

and consequently $\delta = 1$.

We have $p^2+9pq+27q^2 > 0$ for $pq \neq 0$, and from (12) and (13) $l = p^2+9pq+27q^2$, $d^2 = 2pq+9q^2$ and $a^2 = p^2-27q^2$ with $(p, 3) = 1$, $(p, q) = (a, q) = (a, p) = 1$.

In order to solve the last equation, we set $p/q = x$, $q/a = y$. Thus $x^2-27y^2 = 1$. Taking $x = my-1$ we obtain all rational solutions of this equations as follows:

$$x = \frac{p_1^2+27q_1^2}{p_1^2-27q_1^2}, \quad y = \frac{2p_1q_1}{p_1^2-27q_1^2} \tag{14}$$

where $p_1q_1 \neq 0$ and $(p_1, q_1) = 1$ (we exclude $y = 0$ because $q \neq 0$). Hence

$$\frac{p}{a} = \frac{p_1^2+27q_1^2}{p_1^2-27q_1^2}, \quad \frac{q}{a} = \frac{2p_1q_1}{p_1^2-27q_1^2}. \tag{15}$$

On account of $(p, a) = (q, a) = 1$ we have

$$(2p_1q_1,\ p_1^2-27q^2) = (p_1^2+27q_1^2,\ p_1^2-27q_1^2) = \delta_1.$$

By setting $p = 3^s p_1'$, $(p_1', 3) = 1$ we find as in (13) that $s = 0$ or $s \geqslant 3$.

1. If $s = 0$ then $(2p_1q_1,\ p_1^2-27q_1^2) = (2,\ p_1^2-27q_1^2)$ and $\delta_1 = 1$ or $\delta_1 = 2$.

If $\delta_1 = 1$ we have an alternative:

$$\begin{cases} a = p_1^2-27q_1^2, \\ p = p_1^2+27q_1^2, \\ q = 2p_1q_1, \end{cases} \quad \text{or} \quad \begin{cases} a = -(p_1^2-27q_1^2), \\ p = -(p_1^2+27q_1^2), \\ q = -2p_1q_1. \end{cases}$$

In both cases

$$d = 2pq+9q^2 = 4p_1q_1(p_1^2+9p_1q_1+27q_1^2)$$

with $(p_1,\ p_1^2+9p_1q_1+27q_1^2) = 1$ and $(q_1,\ p_1^2+9p_1q_1+27q_1^2) = 1$.

Hence $p_1 = p_2^2$, $q_1 = q_2^2$, $d^2 = 4p_2^2q_2^2(p_2^4+9p_2^2q_2^2+27q_2^4)$ and $p_2^4+9p_2^2q_2^2+27q_2^4 = d_1^2 < d^2 < l^2$. Thus we obtain a solution of the equation (1) in smaller natural numbers, which is contradictory.

If $\delta_1 = 2$ then

$$\begin{cases} 2a = p_1^2-27q_1^2, \\ 2p = p_1^2+27q_1^2, \\ 2q = 2p_1q_1, \end{cases} \quad \text{or} \quad \begin{cases} 2a = -(p_1^2-27q_1^2), \\ 2p = -(p_1^2+27q_1^2), \\ 2q = -2p_1q_1, \end{cases}$$

In both cases $d^2 = 2pq+9q^2 = p_1q_1(p_1^2+9p_1q_1+27q_1^2)$, and we obtain, as above, $p_1 = p_2^2$, $q_1 = q_2^2$ and $p_2^4+9p_2^2q_2^2+27q_2^4 = d_1^2 < d^2 < l^2$, which is contradictory.

2. If $s \geqslant 3$ then $p_1 = 27p_1'$, $(q_1, 3) = 1$ and we have

$$\frac{p}{a} = \frac{27p_1'^2+q_1^2}{27p_1'^2-q_1^2}, \quad \frac{q}{a} = \frac{2p_1'q_1}{27p_1'^2-q_1^2},$$

i. e. a system of the form (15). Hence

$$(2p_1'q_1,\ 27p_1'^2-q_1^2) = (27p_1'^2+q_1^2,\ 27p_1'^2-q_1^2) = \delta_1$$

where $\delta_1 = 1$ or $\delta_1 = 2$ and we obtain inferences, as in the case of $s = 0$ in (15).

The above discussion proves that the equation (1) has no integral solutions if $(a, 3) = 1$. Hence we have Theorem 1.

It follows from I that there exist no integral solutions of the equation $x^3+y^3 = 2z^3$ if $x \neq y$, $xyz \neq 0$. Thus we have proved Theorem 2.

The method of indefinite descent used in Part II leads likewise to the solution of the equations $k^4 \pm 3k^2b^2+3b^4 = c^2$ to which the equations $a^3 \pm b^6 = c^2$ can be brought [2]).

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[2]) See A. Wakulicz, *O bokach sześciennych trójkątów pitagorejskich* (in Polish), Zeszyty Naukowe Wyższej Szkoły Pedagogicznej, Katowice 1957, in press.