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REMARKS ON INVARIANT FUNCTIONS IN MARKOV PROCESSES

BY

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1. Let X be a finite or denumerable set. By $\langle\Omega(X), B_{\Omega(X)}, P\rangle$ we shall denote the stochastic process satisfying the following conditions:

1° The sample functions $\omega\in\Omega(X)$ are X-valued step functions defined for $t\geqslant 0$;

2° $B_{\Omega(X)}$ is the Borel field of subsets of $\Omega(X)$ generated by the class of sets of the form

(1) $$A(t,x)=\{\omega:\omega(t)=x\}\qquad(t\geqslant 0,\ x\in X).$$

3° P is a probability measure in $B_{\Omega(X)}$.

4° There is a continuous function $p(t,x,y)$ of the variable t $(t\geqslant 0;\ x,y\in X)$ satisfying the following conditions:

(α) $p(t,x,y)\geqslant 0$,

(β) $\sum_{y\in X} p(t,x,y)=1$,

(γ) $p(t_1+t_2,x,y)=\sum_{z\in X} p(t_1,x,z)p(t_2,z,y)$,

(δ) $P\left(\bigcap_{i=0}^{n}\{\omega:\omega(t_j)=x_j\}\right)=P(\{\omega:\omega(0)=x_0\})\prod_{j=1}^{n}p(t_j-t_{j-1},x_{j-1},x_j)$ for $0=t_0<t_1<\ldots<t_n$.

In the present note a stochastic process $\langle\Omega(X), B_{\Omega(X)}, P\rangle$ is called briefly a *Markov process*, and a function $p(t,x,y)$ is called a *transition probability*.

Let us consider a Markov process $\langle\Omega(X), B_{\Omega(X)}, P\rangle$. In view of a theorem of Lévy ([2], p. 362) for every $x\in X$ exists the limit

(2) $$Q(x)=\lim_{t\to\infty}P(\{\omega:\omega(t)=x\}).$$

The function $Q(x)$ is called the *limit distribution* of the process $\langle\Omega(X), B_{\Omega(X)}, P\rangle$.

Obviously $\sum_{x\in X} Q(x) \leqslant 1$. Moreover, from the conditions (γ) and (δ) it follows that the equality

$$Q(y) = \sum_{x\in X} p(t, x, y) Q(x) \tag{3}$$

is true.

By $B_{\Omega(X)}(\tau)$ $(-\infty < \tau < \infty)$ we shall denote the Borel field of subsets of $\Omega(x)$ generated by the class of sets of the form (1) for which $t \geqslant \max(0, \tau)$. Obviously, for $\tau \leqslant 0$ the following equality holds: $B_{\Omega(X)}(\tau) = B_{\Omega(X)}$.

We shall define the transformation T_τ $(-\infty < \tau < \infty)$ for the sets of the form (1) with $t \geqslant \tau$ by the formula

$$T_\tau\{\omega : \omega(t) = x\} = \{\omega : \omega(t-\tau) = x\}.$$

Putting

$$T_\tau \bigcup_j E_j = \bigcup_j T_\tau E_j, \qquad T_\tau E' = (T_\tau E)',$$

we obtain the definition of transformation T_τ for each set belonging to $B_{\Omega(X)}(\tau)$.

Let $P(E \mid \omega(t) = x)$ $(E \in B_{\Omega(X)}(t), x \in X)$ be the conditional probability of E if $\omega(t) = x$ (cf. *e. g.* [1], p. 258). It is easy to prove that the equality

$$P(T_\tau E \mid \omega(t) = x) = P(E \mid \omega(t+\tau) = x) \qquad (t \geqslant \tau;\ E \in B_{\Omega(X)}(\tau)) \tag{4}$$

is true.

A set E is called *invariant under the transformation* T_τ if $E \in B_{\Omega(X)}(\tau)$ and $T_\tau E = E$.

A real-valued function f defined on $\Omega(X)$ is called an *invariant function* of the process $\langle \Omega(X), B_{\Omega(X)}, P\rangle$ if for every Borel set U of real numbers the set $\{\omega : f(\omega) \in U\}$ is invariant under all transformations T_τ $(-\infty < \tau < \infty)$.

The purpose of this note is to examine the power of sets A for which $P(\{\omega : f(\omega) \in A\}) = 1$, where f is an invariant function of a Markov process. This problem has been raised by C. Ryll-Nardzewski.

II. From a theorem of Doob ([1], pp. 460, 511) we obtain the following proposition:

If the equality

$$\sum_{x\in X} Q(x) = 1 \tag{5}$$

is satisfied for the limit distribution of a Markov process, then each invariant function of this process assumes essentially $\overline{\overline{X}}$ *values at the most* [1]).

[1]) *i. e.*, for each invariant function f there exists a set A, such that $\overline{\overline{A}} \leqslant \overline{\overline{X}}$ and $P(\{\omega : f(\omega) \in A\}) = 1$ (where $\overline{\overline{B}}$ denotes the power of the set B).

Proof. Let $\langle \Omega(X), B_{\Omega(X)}, P\rangle$ be a Markov process with transition probabilities $p(t, x, y)$. Let us suppose that equality (5) holds. We shall introduce the following probability measure:

$$P^*(E) = \sum_{x\in X} P(E \mid \omega(0) = x)\, Q(x). \tag{6}$$

It is easy to verify that $\langle \Omega(X), B_{\Omega(X)}, P^*\rangle$ is also a Markov process with transition probabilities $p(t, x, y)$. Moreover, $\langle \Omega(X), B_{\Omega(X)}, P^*\rangle$ is a stationary Markov process, *i. e.*, for each $t \geqslant 0$ and $y \in X$ the equality

$$P^*(\{\omega : \omega(t) = y\}) = P^*(\{\omega : \omega(0) = y\}) \tag{7}$$

is true. In fact, definition (6) implies that the following equalities hold:

$$P^*(\{\omega : \omega(t) = y\}) = \sum_{x\in X} p(t, x, y) Q(x), \qquad P^*(\{\omega : \omega(0) = y\}) = Q(y).$$

Hence, according to (3), we obtain formula (7).

Let f be an invariant function of the process $\langle \Omega(X), B_{\Omega(X)}, P\rangle$. Obviously, f is also an invariant function of the process $\langle \Omega(X), B_{\Omega(X)}, P^*\rangle$. Since for each Borel set A of real numbers $\{\omega : f(\omega) \in A\} \in B_{\Omega(X)}(\tau)$ for every τ, formula (4) implies

$$\begin{aligned} P(\{\omega : f(\omega) \in A\} \mid \omega(\tau) = x) &= P(T_\tau\{\omega : f(\omega) \in A\} \mid \omega(\tau) = x) \\ &= P(\{\omega : f(\omega) \in A\} \mid \omega(0) = x). \end{aligned}$$

Consequently, for each A,

$$P(\{\omega : f(\omega) \in A\}) = \sum_{x\in X} P(\{\omega : f(\omega) \in A\} \mid \omega(0) = x)\, P(\{\omega : \omega(\tau) = x\}).$$

Then if $\tau \to \infty$ according to (2) and (6), we obtain

$$P(\{\omega : f(\omega) \in A\}) = P^*(\{\omega : f(\omega) \in A\}). \tag{8}$$

We have proved that f is an invariant function of the stationary Markov process $\langle \Omega(X), B_{\Omega(X)}, P^*\rangle$. According to the theorem of Doob ([1], pp. 460, 511) there is a function f^* measurable relative to the Borel field $\mathfrak{F}$ generated by the class of sets of the form $\{\omega : \omega(0) = x\}$ $(x \in X)$, such that

$$P^*(\{\omega : f(\omega) = f^*(\omega)\}) = 1. \tag{9}$$

It is easy to see that there exist at most $\overline{\overline{X}}$ disjoint sets belonging to $\mathfrak{F}$. There is then a set A of power at most $\overline{\overline{X}}$, such that $P^*(\{\omega : f^*(\omega) \in A\}) = 1$. Consequently, according to (8) and (9), $P(\{\omega : f(\omega) \in A\}) = 1$.

The theorem is thus proved.

III. Let X_0 be an arbitrary finite or denumerable set. We shall prove the following proposition:

For each sequence $q(x)$ $(x \epsilon X_0)$ *of non-negative real numbers satisfying the inequality*

(10) $$\sum_{x \epsilon X_0} q(x) < 1,$$

there is a Markov process $\langle \Omega(X), B_{\Omega(X)}, P\rangle$ *such that*

$$X \supset X_0,$$

$$Q(x) = \begin{cases} q(x) & \text{for} \quad x \epsilon X_0, \\ 0 & \text{for} \quad x \epsilon X - X_0, \end{cases}$$

and there exists an invariant function of this process which assumes essentially non-denumerably many values[2]).

Proof. Let us denote by X^* the set of all systems $\langle i_1, i_2, \ldots, i_n\rangle$, where $i_j = 0$ or 1 $(j = 1, 2, \ldots, n)$ and $n = 1, 2, \ldots$

Without restricting generality we can assume $X^* \cap X_0 = 0$. Put $X = X^* \cup X_0$. Obviously, X is a denumerable set.

We shall define the function $p(t, x, y)$ $(t \geqslant 0; x, y \epsilon X)$ by the following formula:

(11) $$p(t,x,y) = \begin{cases} \dfrac{e^{-t}t^k}{2^k k!} & \text{if} \quad x, y \epsilon X^* \text{ and, for some } n,\ x = \langle i_1, \ldots, i_n\rangle, \\ & \qquad y = \langle i_1, \ldots, i_n, j_1, \ldots, j_k\rangle \quad (k \geqslant 0), \\ 1 & \text{if} \quad x = y \epsilon X_0, \\ 0 & \text{in the other case.} \end{cases}$$

It is easy to verify that the function $p(t, x, y)$ satisfies the conditions (α), (β) and (γ).

Let us write

(12) $$p(x) = \begin{cases} q(x) & \text{if} \quad x \epsilon X_0, \\ 1 - \sum\limits_{y \epsilon X_0} q(x) & \text{if} \quad x = \langle 1\rangle, \\ 0 & \text{if} \quad x \epsilon X^* \text{ and } x \neq \langle 1\rangle. \end{cases}$$

Obviously, the following relations hold:

$$p(x) = 0, \qquad \sum_{x \epsilon X} p(x) = 1.$$

[2]) A function f *assumes essentially non-denumerably many values*, if for each denumerable set A the inequality $P(\{\omega : f(\omega) \epsilon A\}) < 1$ is true.

Considering the cited properties of the functions $p(t, x, y)$ and $p(x)$, it follows from the well-known theorem of Kolmogorov ([3], III, § 4) that there is a stochastic process $\langle \Omega, B_\Omega, P\rangle$ satisfying the following conditions:

(i) the sample functions $\omega \epsilon \Omega$ are X-valued functions defined for $t \geqslant 0$;

(ii) B_Ω is the Borel field of subsets of Ω generated by the class of sets of the form (1);

(iii) P is the probability measure in B_Ω satisfying (δ) and

(13) $$P(\{\omega : \omega(0) = x\}) = p(x) \qquad (x \epsilon X).$$

Formula (1) implies

(14) $$p(t, x, x) = \begin{cases} e^{-t} & \text{for} \quad x \epsilon X^*, \\ 1 & \text{for} \quad x \epsilon X_0. \end{cases}$$

Consequently $\lim\limits_{t \to 0+} p(t, x, x) = 1$ uniformly in x. Hence, according to theorems of Doeblin (cf. *e. g.* Doob [1], pp. 57, 266), we can assume that the sample functions $\omega \epsilon \Omega$ are step functions.

Formula (14) implies

$$\lim_{t \to 0+} \frac{1 - p(t, x, x)}{t} = \begin{cases} 1 & \text{for} \quad x \epsilon X^*, \\ 0 & \text{for} \quad x \epsilon X_0. \end{cases}$$

Therefore, in view of a theorem of Doob ([1], p. 260, 261), we obtain the decomposition $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$, where the sample functions $\omega \epsilon \Omega_1$ are constants, $\omega(t) \equiv x \epsilon X_0$, the sample functions $\omega \epsilon \Omega_2$ are X^*-valued step functions with infinitely many jumps and $P(\Omega_3) = 0$.

Let $\tau_1(\omega) < \tau_2(\omega) < \ldots$ be the sequence of all jump points of a sample function $\omega \epsilon \Omega_2$ and

$$\omega(t) = \begin{cases} z_1(\omega) & \text{for} \quad 0 < t < \tau_1(\omega), \\ z_k(\omega) & \text{for} \quad \tau_{k-1}(\omega) < t < \tau_k(\omega), \quad k = 2, 3, \ldots \end{cases}$$

Obviously, $z_k(\omega)$ $(k = 1, 2, \ldots)$ are X^*-valued measurable functions. Let Ω_0 denote the set of all sample functions $\omega \epsilon \Omega_2$ satisfying for each k the following condition: if $z_k(\omega) = \langle i_1(\omega), \ldots, i_{n(\omega)}(\omega)\rangle$, then $z_{k+1}(\omega) = \langle i_1(\omega), \ldots, i_{n(\omega)}(\omega), j_1(\omega), \ldots, j_{r(\omega)}(\omega)\rangle$.

Let $\{t_j\}$ be the sequence of all positive rational numbers. Then the following inclusion holds:

$$\Omega_2 - \Omega_0 \subset \bigcup_{t_j < t_k} \bigcup_{*} \{\omega : \omega(t_j) = \langle i_1, \ldots, i_n\rangle, \omega(t_k) = \langle j_1, \ldots, j_m\rangle\},$$

where $\bigcup_*$ run over all systems $\langle i_1, \ldots, i_n\rangle \neq \langle j_1, \ldots, j_m\rangle$ satisfying the condition $n > m$ or $n \leqslant m$ and $j_k \neq i_k$ for some $k \leqslant n$. This inclusion implies the inequality

$$P(\Omega_2-\Omega_0) \leqslant \sum_{t_j<t_k} \sum_* p(t_k-t_j, \langle i_1, \ldots, i_n\rangle, \langle j_1, \ldots, j_m\rangle) P(\{\omega:\omega(t_j) = \langle i_1, \ldots, i_n\rangle\}).$$

Hence, according to (11), $P(\Omega_2-\Omega_0) = 0$.

Finally, we have $P(\Omega_0 \cup \Omega_1) = 1$. Setting $\Omega(X) = \Omega_0 \cup \Omega_1$ we obtain the Markov process $\langle \Omega(X), B_{\Omega(X)}, P\rangle$.

From the conditions (γ), (δ) and from the definition of the probability measure P follows the formula

$$P(\{\omega:\omega(t) = x\}) = \sum_{y \epsilon X_0} p(t, y, x) p(y).$$

Hence in, view of (11) and (12),

$$(15)\qquad P(\{\omega:\omega(t)= x\}) = \begin{cases} \left(1-\sum\limits_{y \epsilon X_0} q(y)\right) \dfrac{e^{-t}t^{n-1}}{2^{n-1}(n-1)!} & \text{if} \quad x = \langle 1, i_2, \ldots, i_n\rangle, \\ q(x) & \text{if} \quad x \epsilon X_0, \\ 0 & \text{if} \quad x = \langle 0, i_2, \ldots, i_n\rangle. \end{cases}$$

Consequently

$$Q(x) = \lim_{t\to\infty} P(\{\omega:\omega(t) = x\}) = \begin{cases} q(x) & \text{for} \quad x \epsilon X_0, \\ 0 & \text{for} \quad x \epsilon X-X_0. \end{cases}$$

Moreover, for each $\omega \epsilon \Omega_0$ there exists a zero-one sequence $r_1(\omega)$, $r_2(\omega), \ldots$ and a sequence of integers $n_1(\omega) < n_2(\omega) < \ldots$ such that

$$z_k(\omega) = \langle r_1(\omega), \ldots, r_{n_k(\omega)}(\omega)\rangle \qquad (k = 1, 2, \ldots).$$

It is easy to see that $r_k(\omega)$ $(k = 1, 2, \ldots)$ are measurable functions and for each τ the equality

$$\{\omega:\omega \epsilon \Omega_0, r_k(\omega) = i\} = \bigcap_{t_j>\tau} \bigcup_{n=k+1}^{\infty} \bigcup_{j_1,\ldots,j_{k-1},j_{k+1},\ldots,j_n} \{\omega:\omega(t_j) = \langle j_1,\ldots, j_{k-1}, i, j_{k+1}, \ldots, j_n\rangle\}$$

is true. This implies

$$\{\omega:\omega \epsilon \Omega_0, r_k(\omega) = i\} \epsilon B_{\Omega(X)}(\tau) \qquad (i = 0, 1; k = 1, 2, \ldots; -\infty < \tau < \infty)$$

and

$$T_\tau\{\omega:\omega \epsilon \Omega_0, r_k(\omega) = i\} = \{\omega:\omega \epsilon \Omega_0, r_k(\omega) = i\} \qquad (i = 0, 1; k = 1, 2, \ldots, -\infty < \tau < \infty).$$

Thus the sets

$$(16)\qquad \{\omega:\omega \epsilon \Omega_0, r_k(\omega) = i\} \qquad (i = 0, 1; k = 1, 2, \ldots)$$

are invariant under all transformations T_τ $(-\infty < \tau < \infty)$.

For each $\omega \epsilon \Omega(X)$ we shall define

$$(17)\qquad f_0(\omega) = \begin{cases} \sum\limits_{n=1}^{\infty} \dfrac{r_n(\omega)}{3^n} & \text{if} \quad \omega \epsilon \Omega_0, \\ 2 & \text{if} \quad \omega \epsilon \Omega_1. \end{cases}$$

The set of all values of this function is equal to $C \cup \{2\}$ (C denotes the Cantor set).

We shall prove that f_0 is an invariant function of the process $\langle \Omega(X), B_{\Omega(X)}, P\rangle$.

Let I be an interval of the form

$$(18)\qquad \left[\sum_{n=1}^{k} \frac{i_n}{3^n}, \sum_{n=1}^{k} \frac{i_n}{3^n} + \frac{1}{2\cdot 3^k}\right] \qquad (i_n = 0, 1; k = 1, 2, \ldots).$$

Then

$$\{\omega:f_0(\omega) \epsilon I\} = \bigcap_{j=1}^{k} \{\omega:\omega \epsilon \Omega_0, r_j(\omega) = i_j\}.$$

From the invariance of sets (16) it follows that $\{\omega:f_0(\omega) \epsilon I\}$ is invariant under all transformations T_τ $(-\infty < \tau < \infty)$. From the definition of Ω_1 and from (17) it follows that $\{\omega:f_0(\omega) = 2\} = \Omega_1$ is also invariant under all transformations T_τ. Obviously, the Borel field generated by the class of sets (18) and set $\{2\}$ is equal to the field of all Borel subsets of $C \cup \{2\}$. Consequently, for each Borel set U, $\{\omega,:f_0(\omega) \epsilon U\}$ is invariant under all transformations T_τ. We have thus proved that f_0 is the invariant function of the process $\langle \Omega(X), B_{\Omega(X)}, P\rangle$.

Now, we shall prove that f_0 assumes essentially non-denumerably many values.

Let

$$x = \sum_{n=1}^{\infty} \frac{i_n}{3^n} \epsilon C.$$

Then, in view of (17),

$$\{\omega : f_0(\omega) = x\} = \bigcap_{t\geqslant 0} \bigcup_{n=1}^{\infty} \{\omega : \omega(t) = \langle i_1, \ldots, i_n\rangle\}.$$

Hence, according to (15),

$$(\{\omega : f_0(\omega) = x\}) \leqslant \Big(1 - \sum_{y \epsilon X_0} q(y)\Big) \lim_{t\to\infty} \sum_{n=0}^{\infty} \frac{e^{-t}t^n}{2^n n!} = 0. \tag{19}$$

Let A be an arbitrary denumerable set of real numbers. Then

$$\{\omega : f_0(\omega) \epsilon A\} \subset \{\omega : f_0(\omega) = 2\} \cup \bigcup_{x \epsilon A \cap C} \{\omega : f_0(\omega) = x\}.$$

Hence, according to (10), (15), (17) and (19),

$$P(\{\omega : f_0(\omega) \epsilon A\}) \leqslant P(\{\omega : f_0(\omega) = 2\}) = \sum_{x \epsilon X_0} q(x) < 1.$$

Then f_0 assumes essentially non-denumerably many values. The theorem is thus proved.

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