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ON THE INTERSECTION OF A LINEAR SET WITH THE TRANSLATION OF ITS COMPLEMENT

BY

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(a, b) denotes the closed interval $\{x: a \leqslant x \leqslant b\}$ and $[a, b]$ denotes the set of integers which belong to (a, b). The set $[a, b]$ is also called an *interval*. If E is a set of numbers then we denote by E_t the translated set $\{x+t: x \in E\}$. For a finite set S let $|S|$ be the number of elements in S. For a Lebesgue measurable set Z we denote by mZ the measure of Z. We suppose now that X is any measurable subset of an interval $I = (a, b)$, that $Y = I \setminus X$ and that similarly A and B are any complementary subsets of $[1, N]$. It is the purpose of this paper to prove the following theorems:

THEOREM 1. *There exists such an integer n that*

$$|A_n \cap B| \geqslant \frac{N}{5}\left(2-\sqrt{4-10|A|\,|B|/N^2}\right). \tag{1}$$

THEOREM 2. *There exists such a number t that*

$$m(X_t \cap Y) \geqslant \frac{mI}{5}\left(2-\sqrt{4-10mXmY/(mI)^2}\right). \tag{2}$$

Estimations similar to that which we give in Theorem 1 were first considered by P. Erdös and P. Scherk[1]). P. Erdös found that if $|A| = |B|$, then $\max\limits_n M_n > N/8$ where $M_n = |A_n \cap B|$. This was improved by P. Scherk, who obtained $\max\limits_n M_n > N(2-\sqrt{2})/4$. From Theorem 1 follows the stronger result $\max\limits_n M_n > N(4-\sqrt{6})/10$.

Jan Mycielski proved that if X, Y are measurable subsets of the interval $I = (0, 1)$, then for some t

$$m(X_t \cap Y) \geqslant 1-\sqrt{1-mXmY}.$$

[1]) P. Erdös, *Some remarks on number theory*, Riveon Lematematika 9 (1955), p. 45-48.

If X and Y are complementary sets, then this result follows from Theorem 2.

1. We shall prove first that Theorem 1 implies Theorem 2. It is evident that (2) is invariant under affine transformations of $\mathcal{X}\cdot\mathcal{Y}\cdot\mathcal{I}$. Thus it is sufficient to prove Theorem 2 for $I = (0, 1)$.

Let ε be any positive number. If N is sufficiently large, then X can be covered by such a sum X^* of intervals

$$Q^k = \left(\frac{k-1}{N}, \frac{k}{N}\right)$$

that $m(X \dot{-} X^*) < \varepsilon$ [2]). If $Y^* = I \setminus X^*$, then evidently $m(Y \dot{-} Y^*) < \varepsilon$. We put $A = \{k: Q^k \subset X^*\}$, $B = \{k: Q^k \subset Y^*\}$. These sets satisfy the assumptions of Theorem 1, and consequently (1) holds for some n. From (1) and from the obvious equalities $|A| = mX^*\cdot N$, $|B| = mY^*\cdot N$, $|A_n \cap B| = m(X^*_{n/N} \cap Y^*)\cdot N$ follows

$$(3) \qquad m(X^*_t \cap Y^*) \geqslant \tfrac{1}{5}(2-\sqrt{4-10mX^*mY^*}) \quad \text{for} \quad t = n/N.$$

Let us observe now that, since $m(X_t \dot{-} X^*_t) < \varepsilon$ holds for each t, we have

$$(4) \qquad |m(X_t \cap Y) - m(X^*_t \cap Y^*)| < 2\varepsilon$$

by $(X_t \cap Y) \dot{-} (X^*_t \cap Y^*) \subset (X_t \dot{-} X^*_t) \cup (Y \dot{-} Y^*)$. Since ε is arbitrary, we get from (3) and (4)

$$\sup_t m(X_t \cap Y) \geqslant \tfrac{1}{5}(2-\sqrt{4-10mXmY}).$$

In order to obtain (2) we observe that $m(X_t \cap Y)$ is a continuous function of t since, by (4), the functions $m(X^*_t \cap Y^*)$ approximate it uniformly.

2. We shall give a proof of Theorem 1. In this section we reduce this task to the proof that a certain inequality (5) implies another one (6).

One observes easily that $|A_n \cap B|$ is the number of all such pairs $\langle x, y\rangle$ that $x \in A$, $y \in B$ and $x+n = y$. We denote the set of these pairs by D_n and define $M_n = |D_n|$. Since $\bigcup_{|n|<N} D_n$ contains $|A||B|$ elements, we have

$$|A||B| = \sum_{|n|<N} M_n.$$

[2]) $X \dot{-} X^*$ denotes the symmetric difference $(X\setminus X^*) \cup (X^* \setminus X)$.

Thus, for $0 < k < N$,

$$|A||B| = \sum_{|n|<k} M_n + \sum_{|n|\geqslant k} M_n.$$

If we write R_k for $\sum_{|n|\geqslant k} M_n$, then this equality implies

$$(5) \qquad |A||B| \leqslant 2(k-1)\max M_n + R_k.$$

Let us suppose now that from (5) *follows*

$$(6) \qquad \max M_n > \left(\delta + \varphi\left(\frac{1}{N}\right)\right)N,$$

where $\delta = (2-\sqrt{4-10|A||B|/N^2})/5$ *and* $\lim_{x\to 0}\varphi(x) = 0$.

Let q be any positive integer. We shall prove that

$$(7) \qquad \max M_n > \left(\delta + \varphi\left(\frac{1}{Nq}\right)\right)N.$$

We define

$$\bar{A} = \{lq+j: l+1 \in A,\ 1 \leqslant j \leqslant q\}, \qquad \bar{B} = [1, Nq]\setminus \bar{A}.$$

We prove first that, for $\bar{M}_n = |\bar{A}_n \cap \bar{B}|$,

$$(8) \qquad q\max M_n \geqslant \max \bar{M}_n.$$

Let us observe that $\bar{M}_{nq+j} = (q-j)M_n + jM_{n+1}$ for $j = 0, 1, \ldots, q$. Therefore there exists such an l that $\max \bar{M}_n = \bar{M}_{lq} = qM_l$. This implies formula (8).

Inequality (6) can be applied to $\bar{A}$ and $\bar{B}$. We obtain then

$$(9) \qquad \max \bar{M}_n > \left(\bar{\delta} + \varphi\left(\frac{1}{Nq}\right)\right)Nq.$$

But $\bar{\delta} = \delta$ by $|\bar{A}| = q|A|$, $|\bar{B}| = q|B|$. Thus (7) follows from formulae (8) and (9).

Since q can be arbitrarily large we obtain from (7) $\max M_n \geqslant \delta\cdot N$. This implies Theorem 1.

3. Let us denote by $E \times F$ the Cartesian product of the sets E and F, *i. e.* $E\times F = \{\langle x, y\rangle: x \in E, y \in F\}$. For $W = [0, n]$ we put $\Delta = \{\langle x, y\rangle: x, y, x+y \in W\}$ and $|E, F| = |(E\times F)\cap \Delta|$. Thus $|E, F|$ denotes the number of all points of the integral lattice which lie in the triangle presented in fig. 1 and belong to $E\times F$. We denote $\max M_n$ by d.

Our present aim is to prove the following property of R_k:

3.1. *If* $W = [0, N-k-1]$, *then there exist such sets* U, V, U^*, V^* *that*

(10) $$U \cup V = U^* \cup V^* = W, \quad U \cap V = U^* \cap V^* = \emptyset \text{ [3]}$$

and

(11) $$\big||U|+|U^*|-|W|\big| \leqslant d$$

and

(12) $$R_k = |U, U^*|+|V, V^*|.$$

Proof. We define

$$\overline{U} = A_k \cap W_{k+1}, \qquad \overline{V} = B_k \cap W_{k+1},$$
$$\overline{U}^* = B \cap W_{k+1}, \qquad \overline{V}^* = A \cap W_{k+1}.$$

Then

(13) $$\overline{U} \cup \overline{V} = \overline{U}^* \cup \overline{V}^* = W_{k+1}, \quad \overline{U} \cap \overline{V} = \overline{U}^* \cap \overline{V}^* = \emptyset$$

and

(14) $$|\overline{U}|+|\overline{U}^*| \geqslant |W|-d, \quad |\overline{V}|+|\overline{V}^*| \geqslant |W|-d.$$

Equalities (13) are obvious. Let us prove (14). By $|A_k \cap B| \leqslant d$ and $(A_k \cap B) \subset W_{k+1}$ we obtain $|\overline{U} \cap \overline{U}^*| \leqslant d$. Thus, by (13), $|\overline{V}|+|\overline{V}^*| \geqslant |\overline{V} \cup \overline{V}^*| \geqslant |W|-d$.

Similarly from $|B_k \cap A| = |A_{-k} \cap B| \leqslant d$ and (13) follows the first inequality in (14).

Since (13) implies $|\overline{U}|+|\overline{V}| = |\overline{U}^*|+|\overline{V}^*| = |W|$, we have, by (14),

(15) $$\big||\overline{U}|+|\overline{U}^*|-|W|\big| \leqslant d.$$

Let us now prove that for $\overline{\Delta} = \{\langle x, y\rangle : x \leqslant y,\ x, y \in W_{k+1}\}$

(16) $$R_k = \big|\{(\overline{U} \times \overline{U}^*) \cup (\overline{V} \times \overline{V}^*)\} \cap \Delta\big|.$$

This follows from the equalities

$$|(\overline{U} \times \overline{U}^*) \cap \overline{\Delta}| = \sum_{n \geqslant k} M_n, \quad |(\overline{V} \times \overline{V}^*) \cap \Delta| = \sum_{n \leqslant -k} M_n.$$

We shall prove the first of them. We observe that $\langle x, y\rangle \in (\overline{U} \times \overline{U}^*) \cap \overline{\Delta}$ means that $x-k \in A$, $y \in B$ and $x \leqslant y$. If $x' = x-k$, then this condition

[3] $\emptyset$ denotes the empty set.

is equivalent to $x' \in A$, $y \in B$, $x'+k \leqslant y$ and this means that $\langle x', y\rangle \in \bigcup_{n \geqslant k} D_n$. The equality follows from $|D_n| = M_n$. The proof of the second equality is analogous.

We consider now a transformation τ of the space of all pairs $\langle x, y\rangle$ defined by $\tau\langle x, y\rangle = \langle x-k-1, N-y\rangle$. We define $U = \overline{U}_{-k-1}$, $V = \overline{V}_{-k-1}$, $U^* = \{N-y : y \in \overline{U}^*\}$, $V^* = \{N-y : y \in \overline{V}^*\}$. Since then $\tau(\overline{U} \times \overline{U}^*) = U \times U^*$, $\tau(\overline{V} \times \overline{V}^*) = V \times V^*$ and $\tau(\overline{\Delta}) = \Delta$, we obtain 3.1 from (13), (15) and (16).

4. In this section we shall prove a lemma which will be applied later.

We consider the function $|S, S^*|+|T, T^*|$, where $W = [0, n]$. For $0 \leqslant s, s^* \leqslant n+1$ we denote by $\mu(s, s^*)$ the conditional maximum of this function where the conditions are

(17) $$S \cup T = S^* \cup T^* = W, \quad S \cap T = S^* \cap T^* = \emptyset$$

and

(18) $$|S| = s, \quad |S^*| = s^* \quad \text{or} \quad |T| = s, \quad |T^*| = s^*.$$

For any intervals $L = [a, b]$ and $Q = [c, d]$ let us write $L < Q$ if $a \leqslant c$ and $b \leqslant d$ or if one of them is empty.

LEMMA 1. *If* $0 \leqslant s$, $s^* \leqslant n+1$, *then there exist such disjoint intervals* Φ, Ψ, Ω *and such disjoint intervals* Φ^*, Ψ^*, Ω^* *that* (fig. 2)

$$\Phi \cup \Psi \cup \Omega = \Phi^* \cup \Psi^* \cup \Omega^* = W,$$
$$\Phi < \Psi < \Omega, \quad \Psi^* < \Phi^* < \Omega^*,$$
$$|\Omega^*| \leqslant |\Phi| \leqslant |\Omega^*|+|\Phi^*| \leqslant |\Phi|+|\Psi| \leqslant |W|,$$

and if we define S, S^* *by*

(19) $$S = \Phi \cup \Omega, \quad S^* = \Phi^* \quad \textit{for} \quad \Omega \neq \emptyset,$$
$$\textit{and} \quad S = \Psi, \quad S^* = \Psi^* \cup \Omega^* \quad \textit{for} \quad \Omega = \emptyset$$

and T, T^* *by* (17), *then* (18) *and*

(20) $$|S, S^*|+|T, T^*| = \mu(s, s^*)$$

hold.

COROLLARY OF LEMMA 1. *We have*

(21) $$\mu(s, s^*) = |\Phi||\Phi^*|+|\Psi||\Psi^*| - \tfrac{1}{2}[(|\Phi|-|\Omega^*|)(|\Phi|-|\Omega^*|-1)+ +(|\Phi|+|\Psi|-|\Phi^*|-|\Omega^*|)(|\Phi|+|\Psi|-|\Phi^*|-|\Omega^*|-1)].$$

Proof of the Corollary. Defining S, S^* by (19) and then T, T^* by (17) we find that S, S^*, T, T^* are sums of the intervals $\Phi, \Psi, \Omega, \Phi^*, \Psi^*, \Omega^*$. Substituting these sums in (20) and observing how $|E, F|$ was defined (fig. 1) we easily obtain (21) (fig. 2).

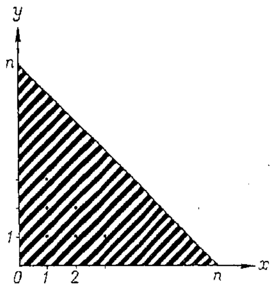

Fig. 1

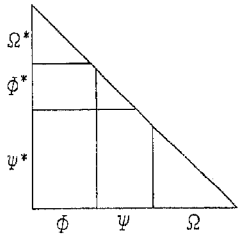

Fig. 2

Proof of Lemma 1. Conditions (17), (18) and (20) are invariant under simultaneous transpositions of S with T and S^* with T^*. Thus the class $\mathcal{K}$ of such quadruples $\{S, S^*, T, T^*\}$ that $n \in S$ and (17), (18), (20) hold is not empty. In the following let us denote by S, S^*, T, T^* such sets that $\{S, S^*, T, T^*\} \in \mathcal{K}$ and $\sigma(S, S^*, T, T^*) \overset{\text{def}}{=} \sum_{x \in S} x$ attains on them its maximal value in $\mathcal{K}$. We shall prove that for these sets it is possible to find intervals $\Phi, \Psi, \Omega, \Phi^*, \Psi^*, \Omega^*$ which have the properties mentioned in Lemma 1 and are such that (19) holds. By this Lemma 1 will be proved.

First we shall show properties 4.1-4.8 of S, S^*, T, T^*. Let us denote by E any of the sets S, S^*, T, T^* and by E^*, D, D^* the remaining three, but in such a way that, with this notation, in the table

$$\begin{array}{c|c} S & T \\ \hline S^* & T^* \end{array}$$

E, E^* will stand in the same column and E, D will stand in the same line.

An interval L will be called *maximal* in E if $L \subseteq E$ but none of the inclusions $L_1 \subset E$, $L_{-1}, I \subset E$ holds.

We define $p(x) = n - x$ and $pQ = \{p(x) : x \in Q\}$ (fig. 3).

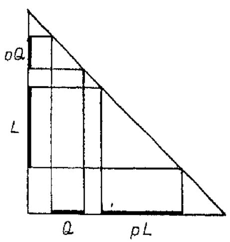

Fig. 3

If L is maximal in E and Q is maximal in E^* and $L \cap pQ \neq \emptyset$, then let us say that L and Q *correspond* to each other or that L (Q) *corresponds to* Q (L).

We shall denote by $\pi(x)$ the propositional function

$$x \in E \quad \text{and} \quad x+1 \in D \quad \text{and} \quad p(x) \in D^*.$$

We shall say that an interval L is *free* if $L_{-1}, L_1 \subset W$.

In the proofs of 4.1 and 4.7 we shall restrict ourselves to one of the possible substitutions of E, E^*, D, D^* for S, S, T^*, T^*. For other substitutions the proofs are analogous.

4.1. *$\pi(x)$ holds for none x.*

Proof. We set $E = S$ and suppose that then $\pi(x)$ holds for some x, *i. e.*

$$x \in S \quad \text{and} \quad x+1 \in T \quad \text{and} \quad p(x) \in T^*.$$

Let us define the sets $\bar{S}, \bar{T}$ by

$$\bar{S} = (S \setminus \{x\}) \cup \{x+1\}, \quad \bar{T} = W \setminus \bar{S}.$$

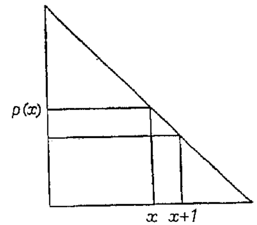

Fig. 4

Evidently (17) and (18) hold if we substitute $\bar{S}, \bar{T}$ for S, T. Consequently

$$|\bar{S}, S^*| + |\bar{T}, T^*| \leqslant \mu(s, s^*). \tag{22}$$

We observe now that

$$|\bar{S}, S^*| - |S, S^*| = |\{x+1\}, S^*| - |\{x\}, S^*| = 0$$

by $p(x) \in T^*$ (see fig. 4). But

$$|\bar{T}, T^*| - |T, T^*| = |\{x\}, T^*| - |\{x+1\}, T^*| = 1.$$

Adding these equalities we obtain

$$|\bar{S}, S^*| + |\bar{T}, T^*| - |S, S^*| - |T, T^*| = 1.$$

This is in contradiction to (20) and (22).

4.2. *If L and Q correspond to each other, then $L < pQ$* (fig. 5).

Proof. We suppose that $L = [a, b] \subset E$ and $Q = [g, h] \subset E^*$. Let us prove that $a \leqslant p(h)$. Indeed, if $p(h) < a$ then $p(g) \geqslant a$ holds by $L \cap pQ \neq \emptyset$ (fig. 6). Thus $p(a-1) \in Q$ and $\pi(a-1)$ holds by

$$a-1 \in D, \quad a \in E \quad \text{and} \quad p(a-1) \in E^*.$$

This is impossible by 4.1. Since a transposition of Q and L does no harm, we find that also $g \leqslant p(b)$. From $a \leqslant p(h)$ and $b \leqslant p(g)$ follows $L < pQ$.

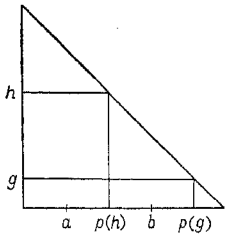

Fig. 5

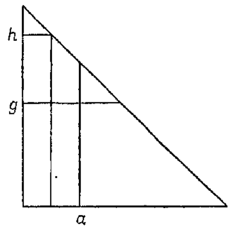

Fig. 6

4.3. *If $L = [a, b]$ is maximal in E and $b+1 \epsilon W$ then $E^* \cap pL \neq \emptyset$.*

Proof. If $E^* \cap pL = \emptyset$, then $pL \subset D^*$. Thus

$$b \epsilon E \quad \text{and} \quad b+1 \epsilon D \quad \text{and} \quad p(b) \epsilon D^*.$$

So $\pi(b)$ holds in contradiction to 4.1.

4.4. *Let L denote an interval maximal in E. Then at most one interval corresponds to L. An interval Q corresponds to L if and only if $E^* \cap pL \neq \emptyset$ and then*

$$pE^* \cap L \subset pQ.$$

Proof. If Q corresponds to L, then $E^* \cap pL \neq \emptyset$ by $Q \subset E^*$ and $Q \cap pL \neq \emptyset$. Conversely, if $E^* \cap pL$ is not empty then this set is contained in a sum of intervals which are maximal in E^* and all correspond to L.

Now, by 4.2, two intervals which correspond to L must intersect. Since they are maximal, only one interval Q corresponds to L. Thus $E^* \cap pL$ is contained in Q. The inclusion follows.

COROLLARY. *If $L_1 \subset W$, then an interval Q corresponds to L.*

This easily follows from 4.3 and 4.4.

4.5. *If L is maximal in E, then there exist such intervals cL, $c'L$ that*

$$L = cL \cup c'L, \quad c'L < cL, \quad c'L \subset pD^*, \quad cL \subset pE^*,$$

and if Q corresponds to L, then $cL = L \cap pQ$.

Proof. We define $cL = L \cap pE^*$, $c'L = L \setminus cL$. If $cL = \emptyset$ then 4.5 holds evidently by 4.4. Form 4.4 it follows also that $cL \neq \emptyset$ holds if and only if an interval Q corresponds to L and that then $cL \subset L \cap pQ$. This inverse inclusion follows from $Q \subset E^*$. Thus we have proved that cL is an interval. From $L < pQ$ it follows that $c'L$ is an interval and that $c'L < cL$ holds.

COROLLARY. *If $L_1 \subset W$, then $cL \neq \emptyset$.*

This easily follows from 4.4 (Corollary) and from 4.5.

4.6. *If $L = [a, b]$ is maximal in E, $L_1 \subset W$ and R denotes that interval which is maximal in D and contains $b+1$ then the interval L^* defined by $pL^* = cL \cup c'R$ corresponds to L.*

Proof. By 4.4 (Corollary) an interval Q corresponds to L. Since from 4.5 follows $pL^* \cap L = pQ \cap L \neq \emptyset$, by 4.4 it is sufficient to prove that L^* is maximal in E^*. We shall do this by showing that pL^* is maximal in pE^*. We shall do this by showing that pL^* is maximal in pE^*.

Obviously $(pL^*)_1 \subset pE^*$ does not hold if $cR \neq \emptyset$. For $cR = \emptyset$ this inclusion is also false since then $R \subset pL^*$ and by 4.5 (Corollary) R_1 is not contained in W.

Now $(pL^*)_{-1} \subset pE^*$ is also false. Namely this inclusion implies $(cL)_{-1} \subset pE^*$, which is impossible since $cL \subset pQ$, $cL < pQ$ by 4.5 and 4.2, and pQ is maximal in pE^*.

4.7. *If L and Q correspond to each other, then they cannot both be free.*

Proof. We set $E = S$ and assume that $L = [a, b] \subset S$ and $Q = [g, h] \subset S^*$ are free. We shall obtain a contradiction. We take the notation

$$(23) \quad \bar{S} = (S \setminus L) \cup L_1, \quad \bar{T} = W \setminus \bar{S}, \quad \bar{S}^* = (S^*/Q) \cup Q_{-1}, \quad \bar{T}^* = W \setminus \bar{S}^*.$$

Let us prove that

$$(24) \qquad |\bar{S}, \bar{S}^*| + |\bar{T}, \bar{T}^*| - \mu(s, s^*) = |L| - |Q| + 2\varepsilon,$$

where $\varepsilon \geqslant 0$. From 4.2 follows $L < pQ$. Thus, by 4.5 (fig. 7),

$$c'L = L \setminus pQ = [a, p(h)-1] \subset pT^*, \qquad c'Q = Q \setminus pL = [g, p(b)-1] \subset pT.$$

This implies

$$(25) \quad [h+1, p(a)] \subset T^*, \qquad [b+1, p(g)] \subset T.$$

Consequently

$$|\bar{S}, \bar{S}^*| - |S, S^*| = |L_1, Q_{-1}| - |L, Q| + \varepsilon,$$

where $\varepsilon = 1$ if $p(g-1) \epsilon S$ and $\varepsilon = 0$ otherwise. Since evidently $|L_1, Q_{-1}| = |L, Q|$ we obtain

$$(26) \qquad |\bar{S}, \bar{S}^*| - |S, S^*| = \varepsilon.$$

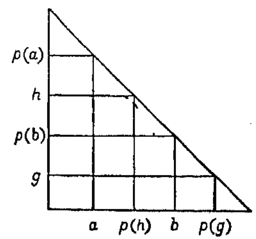

Fig. 7

Let us compute $|\bar{T}, \bar{T}^*| - |T, T^*|$. From (25) follows by $b \leqslant p(g)$ (see fig. 7)

$$(27) \qquad |\bar{T}, T^*| - |T, T^*| = |\{a\}, T^*| - |\{b+1\}, T^*| = p(a) - h.$$

Similarly

(28) $$|\overline{T},\overline{T}^*|-|\overline{T},T^*| = |\overline{T},\{h\}|-|\overline{T},\{g-1\}| = b-p(g)+\varepsilon,$$

which follows from $h \leqslant p(a)$ and (25). Adding (26), (27) and (28) we easily obtain (24).

Let us now transpose in the above considerations each letter of the table

S	T	L	a	b
S^*	T^*	Q	g	h

with that which stands in the same column. We obtain

(29) $$|\overline{S},\overline{S}^*|+|\overline{T},\overline{T}^*|-\mu(s,s^*) = |Q|-|L|+2\eta$$

for $\eta \geqslant 0$ and

(30) $$\overline{S} = (S\backslash L)\cup L_{-1},\quad \overline{T} = W\backslash\overline{S},\quad \overline{S}^* = (\overline{S}^*\backslash Q)\cup Q_1,\quad \overline{T}^* = W\backslash\overline{S}^*.$$

Since $\overline{S},\overline{S}^*,\overline{T},\overline{T}^*$ in (24) and (29) satisfy (17) and (18) we arrive at

$$|L|-|Q|+2\varepsilon \leqslant 0,\qquad |Q|-|L|+2\eta \leqslant 0.$$

It can easily be seen that here equalities must hold. Thus $|\overline{S},\overline{S}^*|+|\overline{T},\overline{T}^*| = \mu(s,s^*)$ is true for (23) and also for (30). But this is in contradiction to our assumption that $\sum_{x\epsilon S} x$ attains its maximum on the sets S, S^*, T, T^*.

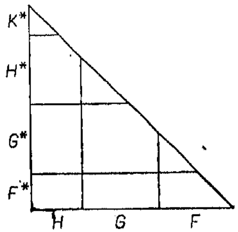

Fig. 8

4.8. *There exist such disjoint intervals F, G, H and such disjoint intervals F^*, G^*, H^*, K^* that* (fig. 8)

$$F\cup G\cup H = F^*\cup G^*\cup H^*\cup K^* = W,$$
$$F^* < G^* < H^* < K^*,\qquad H < G < F,$$
$$|K^*| \leqslant |H| \leqslant |K^*|+|H^*| \leqslant |H|+|G|$$
$$\leqslant |K^*|+|H^*|+|G^*| \leqslant |W|,$$

(31) $$S = F\cup H,\qquad S^* = F^*\cup H^*$$

and if $H \neq \emptyset$, then $F^* = \emptyset$.

Proof. Let F be that interval which contains n and is maximal in S. Evidently there exist such intervals G, H that $H < G < F$, G is empty or maximal in T and H is empty or maximal in S and if $\Gamma = W\backslash(F\cup G\cup H) \neq \emptyset$ then Γ is an interval which contains 0 (fig. 9).

Let us define the intervals F^*, G^*, H^*, K^* by

$$pF^* = cF,\quad pG^* = cG\cup c'F,\quad pH^* = cH\cup c'G,\quad pK^* = c'H,$$

where for $L = \emptyset$ we set $cL = c'L = \emptyset$. Then, by 4.5,

$$cF, c'G, cH \subset pS^*;\qquad c'F, cG, c'H \subset pT^*.$$

Let us prove first that $\Gamma = \emptyset$. Indeed from $\Gamma \neq \emptyset$ follows $F, G, H \neq \emptyset$. Since H is free, we find from 4.6 that H^* corresponds

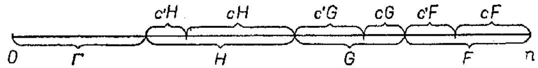

Fig. 9

to H. Since also G is free, it follows from 4.5 (Corollary) that $cG \neq \emptyset$. Thus H^* is free, which is in contradiction to 4.7.

It remains to prove that $H \neq \emptyset$ implies $F^* = \emptyset$. Indeed, if $H \neq \emptyset$ then G is free. By 4.6 G^* corresponds to G. If $F^* \neq \emptyset$ then G^* is free. This is impossible by 4.7.

4.9. We give the last part of the proof of Lemma 1.

If $F^* = \emptyset$ in 4.8, then let us substitute for the letters of the first line of the table

H	H^*	G	G^*	F	K^*
Φ	Φ^*	Ψ	Ψ^*	Ω	Ω^*

those which are under them. Then (31) implies (19) and the other assertions of Lemma 1 obviously hold.

For $H = \emptyset$ (then $K^* = \emptyset$) we substitute in 4.8 for the letters in the first line of the table

G	G^*	F	F^*	H^*
Φ	Φ^*	Ψ	Ψ^*	Ω^*

those which stand under them. It is easy to verify that defining $\Omega = \emptyset$ we obtain the required properties of Φ, Φ^*, Ψ and Ψ^*.

5. We are now in a position to prove that (5) implies (6). Let us define a function $P(u, u^*, v, v^*, \xi)$ by

$$P(u,u^*,v,v^*,\xi) = u(x-u^*-v^*)+vv^*-$$
$$-\tfrac{1}{2}[(v-u^*)(v-u^*-\xi)+(u+v-u^*-v^*)(u+v-u^*-v^*-\xi)].$$

5.1. *There exist such numbers* u, u^*, v, v^* *that for* $1-k/N = x$, $\gamma = d/N$ *and* $\xi = 1/N$ *we have*

(32) $$u^* \leqslant v \leqslant u^*+v^* \leqslant u+v \leqslant x; \quad u, u^*, v, v^* \geqslant 0,$$

(33) $$|u-v^*| \leqslant \gamma,$$

and

(34) $$R_k \leqslant P(u, u^*, v, v^*, \xi)N^2.$$

Proof. We set $W = [0, N-k-1]$. Let M be the conditional maximum of the function $|U, U^*|+|V, V^*|$ under the conditions (10) and (11). From 3.1 follows

(35) $$R_k \leqslant M.$$

We suppose that this maximum is attained on the sets $\overline{U}, \overline{U}^*, \overline{V}, \overline{V}^*$. If $s = |\overline{U}|, s^* = |\overline{U}^*|$, then $S = \overline{U}$, $S^* = \overline{U}^*, T = \overline{V}$, $T^* = \overline{V}^*$ satisfy (17) and (18). Consequently

(36) $$M = \mu(|\overline{U}|, |\overline{U}^*|).$$

Let us consider Lemma 1 for $s = |\overline{U}|, s^* = |\overline{U}^*|$. We find that

(37) $$|\Phi|+|\Omega| = s, \quad |\Phi^*| = s^* \quad \text{or} \quad |\Psi| = s, \quad |\Psi^*|+|\Omega^*| = s^*$$

and

(38) $$|\Phi|+|\Psi|+|\Omega| = |\Phi^*|+|\Psi^*|+|\Omega^*| = |W|.$$

Let us define u, u^*, v, v^* by

$$|\Psi| = Nu, \quad |\Phi| = Nv, \quad |\Omega^*| = Nu^*, \quad |\Phi^*| = Nv^*.$$

Then the conditions (32) hold by Lemma 1. From (11) follows $|s+s^*-N+k| < d$ which implies (33) by (37) and (38). The equalities (21) and (36) imply $M = P(u, u^*, v, v^*, \xi)N^2$. Thus, by (35), we obtain formula (34).

5.2. Let $h(x, \gamma, \xi)$ be the conditional maximum of $P(u, u^*, v, v^*, \xi)$ where (32) and (33) are the conditions. Evidently the function

$$\psi(\xi) = h(a, \gamma, \xi)-h(x, \gamma, 0)$$

satisfies $\lim_{\xi\to 0}\psi(\xi) = 0$ and $\psi(\xi) \geqslant 0$. Computations which we omit here give

$$h(x, \gamma, 0) = \begin{cases} \frac{1}{3}x^2+\frac{1}{2}\gamma^2 & \text{for} \quad 0 \leqslant \gamma \leqslant \frac{1}{3}x, \\ \frac{1}{2}x^2-\frac{1}{4}(x-\gamma)^2 & \text{for} \quad \frac{1}{3}x \leqslant \gamma \leqslant x. \end{cases}$$

If $d \geqslant N/3$ then $d > \delta\cdot N$ by $\delta < 1/3$, and (6) evidently holds. If $d < N/3$, then we define $k = N-3d$. Thus $x = 3\gamma$ and $h(x, \gamma, 0) = 3{,}5\gamma^2$. From (34) follows $R_k \leqslant (3{,}5\gamma^2+\psi(\xi))\cdot N^2$. If we substitute this in (5), we obtain after some simplifications

$$5\gamma^2-4\gamma+2|A||B|/N^2-2\psi(\xi) < 0.$$

Consequently

$$\gamma > \tfrac{1}{5}\left(2-\sqrt{4-10(|A||B|/N^2-\psi(\xi))}\right),$$

by $5|A||B| < 2N^2$ and $\psi(\xi) \geqslant 0$. This implies inequality (6).

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