SOME EXAMPLES OF BOREL SETS

BY

R. SIKORSKI (WARSAW)

The purpose of this paper is to give very simple examples of Borel sets M_α and A_α (lying in the Hilbert cube H) which are exactly of the multiplicative and additive class α respectively ($0 \leqslant \alpha < \Omega$). The definition is by induction.

Let M_0 be a one-point subset of H and $A_0 = H - M_0$. Suppose the sets M_ξ and A_ξ to be defined for all ordinals $\xi < \alpha$. If α is isolated, i. e. $\alpha = \beta + 1$, let

$$M_\alpha = A_\beta \times A_\beta \times A_\beta \times \ldots \subset H^{\aleph_0};$$

if α is a limit ordinal, let

$$M_\alpha = A_1 \times A_2 \times \ldots \times A_\xi \times \ldots = \mathop{\mathrm{P}}_{\xi < \alpha} A_\xi \subset H^{\aleph_0}.$$

Since $H^{\aleph_0}$ is homeomorphic with H, we may treat M_α as a subset of H. Let $A_\alpha = H - M_\alpha$.

It follows immediately from the definition that

(i) *M_α is a Borel set of the multiplicative class α in H; A_α is a Borel set of the additive class α in H.*

The following lemma expresses a remarkable property of the sets M_α and A_α:

(ii) *If X is a metric space and $B \subset X$ is a Borel set of the multiplicative (additive) class α in X, then there exists a continuous mapping φ of X into H such that $\varphi^{-1}(M_\alpha) = B$ (such that $\varphi^{-1}(A_\alpha) = B$).*

The proof is by induction. The case $\alpha = 0$ is obvious. Suppose that α is isolated, i. e. $\alpha = \beta + 1$, and $B \subset X$ is a Borel set of the multiplicative class α, i. e. $B = B_1 B_2 B_3 \ldots$ where $B_n \subset X$ is a Borel set of the additive class β ($n = 1, 2, 3, \ldots$). By the induction hypothesis, there is a continuous mapping φ_n of X into H such that $\varphi_n^{-1}(A_\beta) = B_n$. The continuous mapping $\varphi(x) = \{\varphi_n(x)\} \in H^{\aleph_0} \underset{\text{top}}{=} H$ has the property $\varphi^{-1}(M_\alpha) = B$. If α is a limit ordinal, the proof is similar.

By passage to complements, we prove the lemma for Borel sets B of the additive class α.

(iii) *M_α is not of the additive class α in H; A_α is not of the multiplicative class α in H.*

Let B be a Borel set (in a metric space X) which is of the multiplicative class α, but is not of the additive class α. By (ii), there is a continuous mapping φ of X into H such that $\varphi^{-1}(M_\alpha) = B$. Thus the conjecture that M_α is of the additive class α would imply that B is of the additive class α.

Observe that replacing everywhere the Hilbert cube by the Cantor set or by the set of all irrational numbers, we obtain also sets M_α and A_α satisfying (i) and (iii). Lemma (ii) remains true under the additional hypothesis that X is 0-dimensional.

The defect of the above proof of (iii) is that we used the known fact that there exist Borel sets of arbitrarily high classes. I do not know any direct proof of (iii). I do not know also any solution of the following problem:

P 215. Let A_n be a Borel subset of the additive class α in a metric space X_n, but not of the multiplicative class α in X_n ($n = 1, 2, \ldots$). Prove or disprove that the set $A = A_1 \times A_2 \times \ldots$ (which is, of course, of the multiplicative class $\alpha + 1$) is not of the additive class $\alpha + 1$ in the space $X = X_1 \times X_2 \times \ldots$

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