Proof of Theorem 1. Let A^1 and A^2 be maximal abelian subgroups of the connected compact group G. It is obvious that

$$A^i_{\mathcal{W}} = \bigcap_{m=1}^{\infty} \bigcap_{V_1,\ldots,V_m \in \mathcal{W}} (A^i_{(V_1,\ldots,V_m)})^{\mathcal{W}} \qquad (i = 1, 2).$$

Then on account of Lemma 2 it is easy to see that $A^1_{\mathcal{W}}$ and $A^2_{\mathcal{W}}$ are connected. Also by Lemma 2 for every $V_1, \ldots, V_m \in \mathcal{W}$ the sets

$$H_{V_1,\ldots,V_m} = \{h : h \in G_{(V_1,\ldots,V_m)},\ hA^1_{(V_1,\ldots,V_m)}h^{-1} = A^2_{(V_1,\ldots,V_m)}\}$$

are non empty. Of course, the sets $(H_{V_1,\ldots,V_m})^{\mathcal{W}}$ are compact and $(H_{V_1,\ldots,V_m})^{\mathcal{W}} \cap (H_{V'_1,\ldots,V'_n})^{\mathcal{W}} = (H_{V_1,\ldots,V_m,V'_1,\ldots,V'_n})^{\mathcal{W}}$. Then there exists an

$$h_0 \in \bigcap_{m=1}^{\infty} \bigcap_{V_1,\ldots,V_m \in \mathcal{W}} (H_{V_1,\ldots,V_m})^{\mathcal{W}}.$$

It is clear that $h_0 A^1_{\mathcal{W}} h_0^{-1} = A^2_{\mathcal{W}}$.

Then the groups A^1 and A^2 are also connected and conjugated, q. e. d.

Proof of Theorem 2. By Theorem 1 every element of a connected compact group is contained in a connected compact abelian subgroup of that group. Then by (1) we obtain (8). Now Lemma 1 implies Theorem 2, q. e. d.

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ON THE IMBEDDING OF TOPOLOGICAL GROUPS INTO CONNECTED TOPOLOGICAL GROUPS

BY

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Topological groups arcwise and locally arcwise connected are called *c-groups*. Speaking about subgroups of topological groups we do not suppose that they are closed.

It is the purpose of this paper to prove the following theorem and to make some remarks and pose certain problems connected with it.

Every topological group G is a closed subgroup of a c-group G^*.

Moreover G^* can be such that:

(i) $\operatorname{card} G^* = 2^{\aleph_0} + \operatorname{card} G$;

(ii) *every finitely generated subgroup of G^* is isomorphic to a subgroup of a finite direct product $G \times G \times \ldots \times G$.*

(iii) *G^* has such a normal c-subgroup N (not closed), that every $s \in G^*$ admits the decompositions $s = gh_1$, $s = h_2 g$ where $g \in G$, $h_1, h_2 \in N$, and these decompositions are unique.*

Proof. Let f be a function defined over the half open interval $\langle 0,1)$ with values in the group G for which there exists a finite partition $\langle 0,1) = \langle a_0, a_1) \cup \langle a_1, a_2) \cup \ldots \cup \langle a_n, a_{n+1})$ $(0 = a_0 < a_1 < \ldots < a_{n+1} = 1)$ such that f is constant over each interval $\langle a_i, a_{i+1})$.

Let G^* be the set of all such functions. We introduce the group-operations in G^* by putting

$$[f = gh^{-1}] \equiv [f(x) = g(x)\big(h(x)\big)^{-1} \text{ for every } x \in \langle 0, 1)].$$

Clearly, G^* satisfies (i) and (ii).

It is easy to see that G^* becomes a topological group if we define the open neighbourhoods of a function $f \in G^*$ by

$$\{h : |\{x : h(x) \notin Vf(x)\}| < \varepsilon\} \tag{1}$$

where V is any open neighbourhood of the unity in G, $|\cdot|$ is the Lebesgue measure over $\langle 0, 1)$ and ε is any positive number.

Then G^* is a c-group, because an arc h_t $(0 \leqslant t \leqslant 1)$ joining two functions f and g can be defined by

$$h_t(x) = \begin{cases} f(x) & \text{for} \quad x \in \langle 0, t), \\ g(x) & \text{for} \quad x \in \langle t, 1), \end{cases}$$

and then the neighbourhoods (1) are also arcwise connected.

G can be identified with the subgroup of all constant functions of G^*. Then the first statement of the theorem is proved.

We put $N = \{f : f \in G^*, f(0) = e\}$ and (iii) is visible, q. e. d.

Examples and problems. 1. If G is a metric group with the metric ϱ, then G^* becomes a metric group if we put

$$\operatorname{dist}(f, g) = \int_0^1 \varrho\big(f(x), g(x)\big)\,dx,$$

and the imbedding of G is an isometry.

2. If G is a two-element cyclic group, then by the theorem and the previous remark G^* *is a metric c-group every element of which is of order* 2 [1]).

Another example of such a group can be obtained as follows: Consider the symmetric difference $\dot{-}$ as a group operation over a field $\boldsymbol{F}$ of measurable sets of a space with a σ-additive, *atom free, bounded* measure. Take the factor group $\boldsymbol{F}/\boldsymbol{F}_0$ where $\boldsymbol{F}_0$ is the subgroup of sets of measure 0. The metric in $\boldsymbol{F}/\boldsymbol{F}_0$ is defined by

$$\varrho(A, B) = \operatorname{mes}(A \dot{-} B).$$

It is known that then $\boldsymbol{F}/\boldsymbol{F}_0$ is complete and convex (see *e. g.* [3], p. 169). Hence $\boldsymbol{F}/\boldsymbol{F}_0$ is a c-group.

3. A group G is said to be *functionally free* if for every monomial of the form

$$\sigma(x_1, \ldots, x_m) = x_{i_1}^{k_1} x_{i_2}^{k_2} \ldots x_{i_n}^{k_n} \qquad (k_j = \pm 1,\ x_{i_j}^{k_j} \neq x_{i_{j+1}}^{-k_{j+1}})$$

there exist such elements $\xi_1, \ldots, \xi_m \in G$ that $\sigma(\xi_1, \ldots, \xi_m) \neq e$.

It has been proved [1] that every connected locally compact group which is functionally free (or only not solvable) contains a free subgroup of the rank $2^{\aleph_0}$. The example which follows shows that local compactness is essential in this theorem:

There exists a functionnally free metric c-group each element of which is of finite order.

[1]) The first example of such a group was given by A. A. Markoff. See *e. g.* [2], p. 51.

In fact, take for G the direct product $\prod_{n=1}^{\infty} S_n$, where S_n is the group of permutations of an n-element set. Then every element of G is of finite order and it is easy to see that G is functionally free. We treat G as a metric group with $\varrho(a, b) = 1$ for $a, b \in G, a \neq b$; then, by remark 1, G^* is a metric c-group and is functionally free. By (ii) every element of G^* is of finite order.

4. Concerning the imbedding into compact or locally compact connected groups note that: 1° a compact group is a subgroup of a Cartesian product of compact connected groups of unitary matrices; 2° a locally compact abelian group which is generated by a compact neighbourhood is a subgroup of a connected locally compact abelian group.

This follows from a well known theorem of Pontriagin (see *e. g.* [4], p. 274), stating that every such group is a direct product of a compact group, a free abelian group of finite rank and a vector group.

An infinite discrete abelian group, every element of which is of finite order, cannot be a subgroup of a locally compact, connected abelian group. Otherwise, by the above-mentioned theorem of Pontriagin, it would be contained in a compact group, which is clearly impossible.

P 214. Is it true that a discrete infinite group, every element of which is of finite order, cannot be imbedded into *any* locally compact connected group?

Note added in proof. We due to Professor Freudenthal the following remark:

If a group S contains an infinite subgroup H such that each two elements of $H-(e)$ are conjugated in S, then S with any topology is not a subgroup of a compact group. It follows from the fact that, if G is compact and V is a neighbourhood of e, then there exists such a neighbourhood W of e that $\bigcup_{t \in G} tWt^{-1} \subset V$.

As an example let us take for S the group of linear substitutions $ax+b$ with rational a, b and for H its subgroup of translations.

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