SOME PROPERTIES OF CONNECTED COMPACT GROUPS

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It is the purpose of this paper*) to prove for general compact groups some theorems known for Lie groups. The derivation of these results is based on the theorem of approximation of general compact groups by Lie groups.

Theorem 1 of this paper gives a solution of a problem of S. Hartman and C. Ryll-Nardzewski ([2], P **162**).

1. The groups considered are in general non abelian and written multiplicatively, their unity being denoted by 1. An element x of such a group G is called *divisible* if for every positive integer m there exists such a $\xi\epsilon G$ that $\xi^m = x$. A group G is called *divisible* if all elements of G are divisible.

Our results are the following:

THEOREM 1. *A maximal abelian subgroup of a connected compact group is connected, and all maximal abelian subgroups are conjugated by inner automorphisms.*

THEOREM 2. *An element x of a compact group is contained in the component of the unity if and only if x is divisible.*

The proofs of these theorems are given in sections 2 and 3.

Now we prove some corollaries.

COROLLARY 1. *Every element of a connected compact group is contained in a connected (maximal) abelian subgroup of this group.*

Proof by Theorem 1.

COROLLARY 2. *A compact group is connected if and only if it is divisible.*

Proof by Theorem 2.

COROLLARY 3. *Every connected locally compact group contains an infinite connected abelian subgroup.*

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Proof. It is known[1]) that a connected locally compact group either is compact or contains a one-parameter group.

COROLLARY 4. *Every connected compact group G is covered by the powers of the elements of any neighbourhood of unity.*

Proof. This was proved [2] for divisible groups. Then, by Corollary 2, it holds for all groups which are compact and connected.

Note (this will be applied in our proofs) that:

(1) *Corollary 2 is known for abelian groups* (*e. g.* see [4], p. 55).

(2) *Theorem 1 is known for Lie groups* (*e. g.* see [1], p. 29).

It is also known that every connected locally compact abelian group is divisible[2]). This theorem fails for non abelian groups. In fact, in the group of real matrices A of dimension 2×2 with $\det A = 1$ no matrix exists such that [3])

$$A^2 = \begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

2. For proving our theorems we need some lemmas (implicitly contained in the theorems).

LEMMA 1. *Every divisible element x of a compact group G is contained in a connected abelian subgroup of G.*

Proof. Since x is divisible there exists such a sequence $x_1, x_2, \ldots \epsilon G$ that $x_1 = x$ and $x_m^{m!} = x_{m-1}$ $(m = 2, 3, \ldots)$. The group X generated by the elements $x_1, x_2, \ldots$ is abelian and divisible. Its closure $\overline{X}$ is compact and abelian. $\overline{X}$ is also divisible because, the functions $f_m(\xi) = \xi^m$ being continuous, we have $f_m(\overline{X}) = \overline{f_m(X)} = \overline{X}$. Then, by (1), $\overline{X}$ is connected, q. e. d.

For the proof and formulation of the next lemma we need the following facts and notation:

It is well known that every neighbourhood V of the unity of a compact group G contains a normal subgroup N_V that G/N_V is a Lie group (see [5]). Consider the group-topological Cartesian product

$$\mathrm{P}(\mathcal{V}) = \mathop{\mathrm{P}}_{V\epsilon\mathcal{V}} G/N_V,$$

[1]) By a theorem of K. Iwasawa [3] (Theorem 6) and the main approximation theorem of H. Yamabe (see [5], p. 175).

[2]) By a theorem of Pontrjagin every such group is a direct product of a vector group and a compact group. Hence this statement follows from (1).

[3]) I am indebted for this example to A. Goetz.

where $\mathcal{V}$ is a family of neighbourhoods of the unity of G. Then $\mathrm{P}(\mathcal{V})$ is a Lie group if $\mathcal{V}$ is finite. Consider the natural homomorphism $\varphi_{\mathcal{V}}: G \to \mathrm{P}(\mathcal{V})$, *i. e.*, the homomorphism defined as follows: if $\mathcal{V} = (V)$ then $\varphi_{\mathcal{V}}$ is the natural homomorphism of G onto G/N_V; for general $\mathcal{V}$ if $x \in G$ and $\varphi_{\mathcal{V}}(x) = f$, then $f(V) = \varphi_{(V)}(x)$ for all $V \in \mathcal{V}$. Of course, $\varphi_{\mathcal{V}}$ is a continuous mapping. We abbreviate the notation by putting $\varphi_{\mathcal{V}}(x) = x_{\mathcal{V}}$ and $\varphi_{\mathcal{V}}(X) = X_{\mathcal{V}}$ for any $x \in G$ and $X \subset G$.

Let $\mathcal{W}$ denote a complete system of neighbourhoods of the unity in G. It is clear that $G_{\mathcal{W}}$ is a topological and algebraical isomorph of G.

If $\mathcal{V}_1 \subset \mathcal{V}_2$ and $A \subset \mathrm{P}(\mathcal{V}_2)$, then $A_{\mathcal{V}_1}$ denotes the projection of A on $\mathrm{P}(\mathcal{V}_1)$; if $B \subset \mathrm{P}(\mathcal{V}_1)$ then $B^{\mathcal{V}_2}$ denotes the intersection of $G_{\mathcal{V}_2}$ and the cylinder in $\mathrm{P}(\mathcal{V}_2)$ with basis B.

Especially, if $\mathcal{V}$ is finite, $\mathcal{V} = (V_1, \ldots, V_m)$, we write $x_{(V_1,\ldots,V_m)}$, $X_{(V_1,\ldots,V_m)}$ etc.

Clearly, if $\mathcal{V} \subset \mathcal{W}$ and $X \subset G$, then $(X_{\mathcal{W}})_{\mathcal{V}} = X_{\mathcal{V}}$.

LEMMA 2. *If G is connected and compact, A^1 and A^2 are maximal abelian subgroups of G, then for every finite system $\mathcal{V} = (V_1, \ldots, V_m) \subset \mathcal{W}$ the groups $A^1_{\mathcal{V}}$ and $A^2_{\mathcal{V}}$ are connected conjugated subgroups of $G_{\mathcal{V}}$.*

Proof. Let B_1 and B_2 be any open sets in $G_{\mathcal{V}}$ containing respectively $A^1_{\mathcal{V}}$ and $A^2_{\mathcal{V}}$.

We put

$$H_{B_1B_2} = \{h: h \in G_{\mathcal{V}},\ hA^1_{\mathcal{V}}h^{-1} \subset \overline{B}_2,\ h^{-1}A^2_{\mathcal{V}}h \subset \overline{B}_1\}^{4)}.$$

We shall prove that

(3) *the sets $H_{B_1B_2}$ are non empty;*

(4) *there exist such connected sets S_i that $A^i_{\mathcal{V}} \subset S_i \subset B_i$ for $i = 1, 2$.*

For this purpose we shall prove the existence of such a finite set $\mathcal{U}$ that $\mathcal{V} \subset \mathcal{U} \subset \mathcal{W}$ and for every maximal abelian subsgroups A^{1*} and A^{2*} of $G_{\mathcal{U}}$ such that $A^1_{\mathcal{U}} \subset A^{1*}$, $A^2_{\mathcal{U}} \subset A^{2*}$ we have $A^{1*}_{\mathcal{V}} \subset B_1$, $A^{2*}_{\mathcal{V}} \subset B_2$.

This will prove (3) and (4). Indeed, the group $G_{\mathcal{U}}$ is then a Lie group and by (2), A^{1*} and A^{2*} are conjugated in $G_{\mathcal{U}}$, which proves (3), and connected, which proves (4) (we put $S_i = A^{i*}_{\mathcal{V}}$).

Then let us construct such a $\mathcal{U}$. We put $C_i = G_{\mathcal{V}} \setminus B_i$. Then C_i are compact and $C_i \subset G_{\mathcal{V}} \setminus A^i_{\mathcal{V}}$.

For every $x \in G$ if $x_{\mathcal{V}} \in C_i$, then $x \notin A^i$ and there exists such an $a \in A^i$ that $ax \neq xa$. There exists also a neighbourhood W of x such that $1 \notin \overline{\{aya^{-1}y^{-1}: y \in W\}}$.

4) $\overline{X}$ denotes the closure of X.

Since $C_i^{\mathcal{W}}$ is compact, we can take a finite system $W_1, \ldots, W_n$ of open subsets of $G_{\mathcal{W}}$ and a system $a_1, \ldots, a_n \in A^i_{\mathcal{W}}$ such that

(5) $$1 \notin \bigcup_{k=1}^{n} \overline{\{a_k y a_k^{-1} y^{-1}: y \in W_k\}}$$

and

(6) $$C_i^{\mathcal{W}} \subset \bigcup_{k=1}^{n} W_k.$$

By (5) there exists a neighbourhood of unity $U \in \mathcal{W}$ such that

(7) $$U \subset G_{\mathcal{W}} \setminus \bigcup_{k=1}^{n} \overline{\{a_k y a_k^{-1} y^{-1}: y \in W_k\}}.$$

We put $\mathcal{U} = (V_1, \ldots, V_m, U)$. We will verify that $\mathcal{U}$ has the required properties.

Let A^{i*} be a maximal abelian subgroup of $G_{\mathcal{U}}$ which contains $A^i_{\mathcal{U}}$. We must prove that $A^{i*}_{\mathcal{V}} \subset B_i$. It is enough to show that for every $c \in C_i^{\mathcal{U}}$ there exists such an $a \in A^i_{\mathcal{U}}$ that $ac \neq ca$. By (6) and (7) for every $c \in C_i^{\mathcal{U}}$ there exist such an $x \in C_i^{\mathcal{W}}$ and $a_k \in A^i_{\mathcal{W}}$ that $x_{\mathcal{U}} = c$ and $a_k x a_k^{-1} x^{-1} \notin U$. But then $(a_k x a_k^{-1} x^{-1})_{(U)} \neq 1$, *i. e.* $ac \neq ca$ for $a = (a_k)_{\mathcal{U}}$.

Then $\mathcal{U}$ has the required properties.

Now on account of (3) and (4) we prove the Lemma. The sets $H_{B_1B_2}$ are compact and, by (3), all finite intersections of these sets are non empty (because $H_{B_1'B_2'} \cap H_{B_1''B_2''} \supset H_{B_1' \cap B_1'' B_2' \cap B_2''}$). Then there exists an $h_0 \in \bigcap_{B_1,B_2} H_{B_1B_2}$. Consequently,

$$h_0 A^1_{\mathcal{V}} h_0^{-1} \subset \bigcap_{B_2} \overline{B}_2 = A^2_{\mathcal{V}} \quad \text{and} \quad h_0^{-1} A^2_{\mathcal{V}} h_0 \subset \bigcap_{B_1} \overline{B}_1 = A^1_{\mathcal{V}}.$$

Then the groups $A^1_{\mathcal{V}}$ and $A^2_{\mathcal{V}}$ are conjugated; they are closed (because A^1, A^2, being maximal, are closed), then, by (4), they are also connected, q. e. d.

3. Now we prove theorems 1 and 2.

Note that theorem 2 can be proved independently of Theorem 1, if we use the following lemma:

(8) *A connected compact group is divisible.*

The proof of this Lemma (for Lie groups it holds by (1) and (2)) is simpler than that of Theorem 1 and can be easily performed by means of the notions introduced in the preceding section.

Proof of Theorem 1. Let A^1 and A^2 be maximal abelian subgroups of the connected compact group G. It is obvious that

$$A^i_{\mathfrak{W}} = \bigcap_{m=1}^{\infty} \bigcap_{V_1,\ldots,V_m \in \mathfrak{W}} (A^i_{(V_1,\ldots,V_m)})^{\mathfrak{W}} \qquad (i = 1, 2).$$

Then on account of Lemma 2 it is easy to see that $A^1_{\mathfrak{W}}$ and $A^2_{\mathfrak{W}}$ are connected. Also by Lemma 2 for every $V_1, \ldots, V_m \in \mathfrak{W}$ the sets

$$H_{V_1,\ldots,V_m} = \{h \colon h \in G_{(V_1,\ldots,V_m)},\ hA^1_{(V_1,\ldots,V_m)}h^{-1} = A^2_{(V_1,\ldots,V_m)}\}$$

are non empty. Of course, the sets $(H_{V_1,\ldots,V_m})^{\mathfrak{W}}$ are compact and $(H_{V_1,\ldots,V_m})^{\mathfrak{W}} \cap (H_{V'_1,\ldots,V'_n})^{\mathfrak{W}} = (H_{V_1,\ldots,V_m,V'_1,\ldots,V'_n})^{\mathfrak{W}}$. Then there exists an

$$h_0 \in \bigcap_{m=1}^{\infty} \bigcap_{V_1,\ldots,V_m \in \mathfrak{W}} (H_{V_1,\ldots,V_m})^{\mathfrak{W}}.$$

It is clear that $h_0 A^1_{\mathfrak{W}} h_0^{-1} = A^2_{\mathfrak{W}}$.

Then the groups A^1 and A^2 are also connected and conjugated, q. e. d.

Proof of Theorem 2. By Theorem 1 every element of a connected compact group is contained in a connected compact abelian subgroup of that group. Then by (1) we obtain (8). Now Lemma 1 implies Theorem 2, q. e. d.

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ON THE IMBEDDING OF TOPOLOGICAL GROUPS INTO CONNECTED TOPOLOGICAL GROUPS

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Topological groups arcwise and locally arcwise connected are called *c-groups*. Speaking about subgroups of topological groups we do not suppose that they are closed.

It is the purpose of this paper to prove the following theorem and to make some remarks and pose certain problems connected with it.

Every topological group G is a closed subgroup of a c-group G^*.

Moreover G^* can be such that:

(i) $\operatorname{card} G^* = 2^{\aleph_0} + \operatorname{card} G$;

(ii) *every finitely generated subgroup of G^* is isomorphic to a subgroup of a finite direct product $G \times G \times \ldots \times G$.*

(iii) *G^* has such a normal c-subgroup N (not closed), that every $s \in G^*$ admits the decompositions $s = gh_1$, $s = h_2 g$ where $g \in G$, $h_1, h_2 \in N$, and these decompositions are unique.*

Proof. Let f be a function defined over the half open interval $\langle 0,1)$ with values in the group G for which there exists a finite partition $\langle 0,1) = \langle a_0, a_1) \cup \langle a_1, a_2) \cup \ldots \cup \langle a_n, a_{n+1})$ $(0 = a_0 < a_1 < \ldots < a_{n+1} = 1)$ such that f is constant over each interval $\langle a_i, a_{i+1})$.

Let G^* be the set of all such functions. We introduce the group-operations in G^* by putting

$$[f = gh^{-1}] \equiv [f(x) = g(x)\big(h(x)\big)^{-1} \text{ for every } x \in \langle 0, 1)].$$

Clearly, G^* satisfies (i) and (ii).

It is easy to see that G^* becomes a topological group if we define the open neighbourhoods of a function $f \in G^*$ by

$$\{h \colon |\{x \colon h(x) \notin Vf(x)\}| < \varepsilon\} \tag{1}$$

where V is any open neighbourhood of the unity in G, $|\cdot|$ is the Lebesgue measure over $\langle 0, 1)$ and ε is any positive number.