THE COMPLETENESS OF THE HOMEOMORPHISMS GROUP OF A COMPLETE SPACE

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Let X be any completely regular space. It admits a uniform structure \([V_e]\) defined by all neighbourhoods of the set \(A = \{(x, x) : x \in X\}\) in the product topology in \(X \times X\) \([2]\).

A filter \([U_e]\) of a space which admits a uniform structure \([V_e]\) is called a Cauchy filter if, for each \(e\) and some \(\tau\), \(U_\tau \times U_\tau \subseteq V_e\). If every Cauchy filter of the space converges, then this space will be called complete (in the uniform structure \([V_e]\)).

Given any topological group \(H\), one can define a uniform structure \([\mathcal{U}_e]\) by saying that \((x, y) \in \mathcal{U}_e\) if \(x \in \mathcal{U}_e \circ y \circ \mathcal{U}_e\), and \(\mathcal{U}_e\) are neighborhoods of the unity in \(H\). A topological group is called complete in the sense of Raikov if it is complete in the structure \([\mathcal{U}_e]\) \([3]\).

Let \(H\) be the homeomorphisms group of \(X\). It is known that the family of sets \(\mathcal{U}_e = \{h : (x, h(x)) \in V_e\}\) for all \(x \in X\) makes \(H\) a topological group, where \(\mathcal{U}_e\) is the system of neighborhoods of the identity transformation of \(H\) \([4]\).

The aim of this note is the proof of the following

**Theorem**. If the space \(X\) is complete (in the maximal uniform structure of all neighborhoods \(V_e\) in \(X \times X\)) and the homeomorphisms group \(H\) of \(X\) is topologized by the system of neighborhoods of the unity \(\mathcal{U}_e\), then \(H\) is complete in the sense of Raikov.

The proof is based on the following two lemmas:

**Lemma 1.** Let \([U_{e}^x]\) be a Cauchy filter in \(H\); then, for each \(e\) there is a \(\tau\) such that \(h, f \in U_\tau\) implies \((h(x), f(x)) \in V_e\) for all \(x \in X\).

**Proof.** Since \([U_{e}^x]\) is a Cauchy filter, \(h, f \in U_\tau\) implies, for each \(x\) and some \(\tau\), \(x \in \mathcal{U}_\tau\) or \(h^{-1}(x) \in \mathcal{U}_\tau\), for all \(x \in X\), thus \((h(x), f(x)) \in V_e\).

1) The completeness of the homeomorphisms group in the \(p\)-topology of a locally compact space was proved by R. Arens [1]. His proof is based on the local compactness of the space and cannot be transferred to our case.
Lemma 2. Let $\{U_{a}\}$ be the set $\{k(x) : k \in K\}$. If $\{U_{a}\}$ is a Cauchy filter in $X$, then $U_{a}$ is a Cauchy filter in $X$ relative to the structure $[V_{a}]$. Moreover, for each $a$, there exists such a $\tau$ that $U_{a} \subset U_{a} \subset V_{a}$ for every $a$.

Indeed, we have $U_{a} \times U_{a} \subset \{f(a) : f \in F_{a}\}$. Thus it remains to apply Lemma 1.

Proof of the theorem. Let $\{U_{a}\}$ be a Cauchy filter in $X$ (relative to the uniform structure $[V_{a}]$). By Lemma 2, $\{U_{a}\}$ is a Cauchy filter in $X$. By the completeness of $X$, $\{U_{a}\}$ converges, say, to $y$. We are going to prove that for the function $g : x \rightarrow y$

1st for each $a$ there exists such a $\tau$ that

(1) $f \in U_{a}$ implies $\{f(a), g(a)\} \in V_{a}$ for every $x \in X$,

2nd $g(x)$ is a homeomorphism of the space $X$.

Proof. 1st. Let any $V_{a}$ be given. There exists such a $V_{a}$ that $V_{a} \subset V_{a}$.

By Lemma 2 for some $\tau$ and for each $x \in X$ we have $U_{a} \times U_{a} \subset V_{a}$.

Consequently $U_{a} \times U_{a} \subset V_{a} \subset V_{a}$.

Since $f(x), g(x) \in U_{a}$, our assertion follows.

The continuity of $g$ is reached by the following simple reasoning:

For each $x$ we take a $\tau$ such that

(2) $V_{a} \times V_{a} \subset V_{a}$.

For each $a$ and some $\mu$

(3) $(x, y) \in V_{a} \times V_{a}$ implies $(f(a), f(y) \in V_{a}$

and by 1st there exists such a $\tau$ that

(4) $f \in U_{a}$ implies $(f(a), g(a)) \in V_{a}$ and $(f(y), g(y)) \in V_{a}$.

Hence, by (3), (3) and (4), $(x, y) \in V_{a}$ implies $(g(a), g(y)) \in V_{a}$.

If $\{U_{a}\}$ is a Cauchy filter relative to the uniform structure $[V_{a}]$ so is also $\{U_{a}\}$. Consequently $\{U_{a}\}$ converges, say, to $y$. The function $g(x) \rightarrow y$, like the function $g$, is continuous, i. e., for each $a$ and some $\tau$

(5) $(x, y) \in V_{a}$ implies $(g(x), g(y)) \in V_{a}$.

By Lemma 2 for each $V_{a}$ there exists such a $U_{a}$ that

(6) $f \in U_{a}$ implies $(f(a), g(a)) \in V_{a}$ for every $x \in X$.

We take any $V_{a}$ and we choose $V_{a}$ from (5) and $U_{a}$, from (6). For $V_{a}$, we take $U_{a}$ from (1). If $f \in U_{a}$, then, by (6),

(7) $(x, y) \in V_{a}$ implies $(g(x), g(y)) \in V_{a}$.

and, by (1), $(f(a), g(a)) \in V_{a}$; consequently, by (5)

(8) $(g'(f(x)), g'(y(x))) \in V_{a}$.

From (7) and (8) we have $(x, y) \in V_{a}$, $(g(x)) \in V_{a}$. In view of the free choice of $V_{a}$, $(g(x)) = x$, or $g' = g^{-1}$, which completes the proof.

References


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