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Reçu par la Rédaction le 28. 1. 1957

THE COMPLETENESS OF THE HOMEOMORPHISMS GROUP OF A COMPLETE SPACE

BY

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Let X be any completely regular space. It admits a uniform structure $\{V_a\}$ defined by all neighbourhoods V_a of the set $\Delta = \{(x, x): x \in X\}$ in the product topology in $X \times X$ [2].

A filter $\{U_\tau\}$ of a space which admits a uniform structure $\{V_a\}$ is called a *Cauchy filter* if, for each a and some τ, $U_\tau \times U_\tau \subset V_a$. If every Cauchy filter of the space converges, then this space will be called *complete* (in the uniform structure $\{V_a\}$).

Given any topological group H, one can define a uniform structure $\{{}^*\mathfrak{V}_a^*\}$ by saying that $(x, y) \in {}^*\mathfrak{V}_a^*$ if $x \in \mathfrak{V}_a y \cap y\mathfrak{V}_a$ and $\mathfrak{V}_a$ are neighbourhoods of the unity in H. A topological group is called *complete in the sense of Raïkov* if it is complete in the structure $\{{}^*\mathfrak{V}^*\}$ [3].

Let H be the homeomorphisms group of X. It is known that the family of sets $\mathfrak{V}_a = \{h: (x, h(x)) \in V_a \text{ for all } x \in V\}$ makes H a topological group, where $\mathfrak{V}_a$ is the system of neighbourhoods of the identity transformation of H [4].

The aim of this note is the proof of the following

THEOREM[1]). *If the space X is complete (in the maximal uniform structure of all neighbourhoods V_a in $X \times X$) and the homeomorphisms group H of X is topologized by the system of neighbourhoods of the unity $\mathfrak{V}_a$, then H is complete in the sense of Raïkov.*

The proof is based on the following two lemmas:

LEMMA 1. *Let $\{U_\tau\}$ be a Cauchy filter in H; then, for each a there is a τ such that $h, f \in U_\tau$ implies $(h(x), f(x)) \in V_a$ for all $x \in X$.*

Proof. Since $\{U_\tau\}$ is a Cauchy filter, $h, f \in U_\tau$ implies, for each a and some τ, $f \in \mathfrak{V}_a h$ or $(x, f(h^{-1}(x))) \in V_a$ for all $x \in X$, thus $(h(x), f(x)) \in V_a$.

[1]) The completeness of the homeomorphisms group in the g-topology of a locally compact space was proved by R. Arens [1]. His proof is based on the local compactness of the space and cannot be transferred to our case.

LEMMA 2. *Let $U_{\tau,x}$ be the set $\{h(x): h \in U_{\tau}\}$. If $\{U_{\tau}\}$ is a Cauchy filter in H, then $U_{\tau,x}$ is a Cauchy filter in X relative to the structure $\{V_{\alpha}\}$. Moreover, for each α, there exists such τ that $U_{\tau,x} \times U_{\tau,x} \subset V_{\alpha}$ for every x.*

Indeed, we have $U_{\tau,x} \times U_{\tau,x} = \{(f(x), h(x)): f, h \in U_{\tau}\}$. Thus it remains to apply Lemma 1.

Proof of the theorem. Let $\{U_{\tau}\}$ be a Cauchy filter in H (relative to the uniform structure $\{{}^*\mathfrak{B}_{\alpha}^*\}$). By Lemma 2, $\{U_{\tau,x}\}$ ist a Cauchy filter in X. By the completeness of X, $\{U_{\tau,x}\}$ converges, say, to y. We are going to prove that for the function $g: x \to y$

1° for each α there exists such a τ that

(1) $f \in U_{\tau}$ implies $(f(x), g(x)) \in V_{\alpha}$ for every $x \in X$,

2° $g(x)$ is a homeomorphism of the space X.

Proof. 1° Let any V_{α} be given. There exists such a V_{γ} that $\overline{V}_{\gamma} \subset V_{\alpha}$. By Lemma 2 for some τ and for each $x \in X$ we have $U_{\tau,x} \times U_{\tau,x} \subset V_{\gamma}$. Consequently $\overline{U}_{\tau,x} \times \overline{U}_{\tau,x} \subset \overline{V}_{\gamma} \subset V_{\alpha}$.

Since $f(x), g(x) \in \overline{U}_{\tau,x}$, our assertion follows.

2° The continuity of g is reached by the following simple reasoning: For each α we take λ such that[2])

$$V_{\lambda} \circ V_{\lambda} \circ V_{\lambda} \subset V_{\alpha}. \tag{2}$$

For each λ and some μ

(3) $(x, y) \in V_{\mu}$ implies $(f(x), f(y)) \in V_{\lambda}$

and by 1° there exists such a τ that

(4) $f \in U_{\tau}$ implies $(f(x), g(x)) \in V_{\lambda}$ and $(f(y), g(y)) \in V_{\lambda}$.

Hence, by (2), (3) and (4), $(x, y) \in V_{\mu}$ implies $(g(x), g(y)) \in V_{\alpha}$.

If $\{U_{\tau}\}$ is a Cauchy filter relative to the uniform structure $\{{}^*\mathfrak{B}_{\alpha}^*\}$ so is also $\{U_{\tau}^{-1}\}$. Consequently $\{U_{\tau,x}^{-1}\}$ converges, say, to y'. The function $g': x \to y'$, like the function g, is continuous, *i. e.*, for each α and some ν

(5) $(x, y) \in V_{\nu}$ implies $(g'(x), g'(y)) \in V_{\alpha}$.

By lemma 2 for each V_{α} there exists such a U_{τ_1} that

(6) $f \in U_{\tau_1}$ implies $(f^{-1}(x), g'(x)) \in V_{\alpha}$ for every $x \in X$.

We take any V_{α} and we choose V_{ν} from (5) and U_{τ_1} from (6). For V_{ν} we take U_{τ} from (1). If $f \in U_{\tau} \cap U_{\tau_1}$, then, by (6),

[2]) By $V_{\alpha} \circ V_{\beta}$ we mean the class of all pairs (x, z) for which there is an element $y \in x$ such that $(x, y) \in V_{\alpha}$ and $(y, z) \in V_{\beta}$.

$$\big(x, g'((f(x))\big) \in V_{\alpha} \tag{7}$$

and, by (1), $(f(x), g(x)) \in V_{\nu}$; consequently, by (5)

$$\big(g'(f(x)), g'(g(x))\big) \in V_{\alpha}. \tag{8}$$

From (7) and (8) we have $\big(x, g'((g(x))\big) \in V_{\alpha} \circ V_{\alpha}$. In view of the free choice of V_{α}, $(g'(x)) = x$, or $g' = g^{-1}$, which completes the proof.

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Reçu par la Rédaction le 5. 11. 1956