Inequality (13) implies $D(0, x) = \Theta$. Then, in view of (11), relation (iv) holds.

From inequality (14) and from relation (iii) we obtain for $\lambda \neq 0$ and $x \in \mathcal{X}$:

$$\|x\| = \left\| \frac{1}{\lambda} (\lambda x) \right\| \leqslant \frac{1}{|\lambda|} \|\lambda x\|.$$

Hence

$$|\lambda| \|x\| \leqslant \|\lambda x\|.$$

Then, according to (14), relation (v) holds. The theorem is thus proved.

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Reçu par la Rédaction le 2. 7. 1956

*ON THE COMPARISON OF TWO PRODUCTION PROCESSES AND THE RULE OF DUALISM**

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INTRODUCTION

A silk ribbon produced by an automaton is an example of a continuous production process. An observer estimates the quality of the product by counting the defects, say, in a given segment of the product. In this example the role of defects may be played by stains or holes. From the number of defects observed the inspector estimates the defectiveness of the product and expresses it by the number of defects per meter of ribbon.

It may happen that not the estimation of defectiveness but the comparison of two production processes is the purpose of examination. In such a case the above example should be replaced by another, namely by two ribbons running parallel with the same speed. The comparison is a statement that the defectiveness of the second ribbon is at least a times greater than the defectiveness of the first ribbon; it is known that such statements can be deduced from observations with a certain "probability". The observation proceeds according to a certain plan. We shall be concerned with two such plans. The first of them, called *classical*, consists in observing the ribbons until the total of defects in both ribbons reaches a prescribed number N. If n and m are the numbers of defects on the first and second ribbon respectively, we have $m+n = N$. The second plan, called *sequential*, consists in observing the ribbons until on the first ribbon the n-th defect appears, where n is prescribed. Once the plan is chosen and the observation taken we compute by a suitable formula the "probability" P of the statement formulated above. This probability depends on three numbers, a, m and n. Our aim is to discuss the methods of defining the "probability" P.

It is worth while to answer the question why we do not estimate the defectiveness of each process separately. We have here an analogy

* Another version of this paper appeared in Polish, see [8].

with the rule applying to any measuring: it is much easier to state directly how many times a given thing is heavier than another, than to weigh both separately and to compare their weights. In our examples the observation (m, n) immediately answers the question concerning the ratio of defectivenesses; when we want to estimate first the defectivenesses themselves, we must also know the lengths of both ribbons.

The comparison of two production processes is, as we shall see, closely connected with the problem of estimating the fraction of red balls in an urn filled with red and black balls.

§ 1. ASSUMPTIONS

In the sequel we shall assume that the distributions of defects along the ribbons are Poisson distributions. More precisely, we shall assume that the defects on each ribbon separately form a homogeneous Poisson stochastic process. It means that there is a constant c such that for every segment $(t, t+h)$ of the ribbon the probability $P_k(h)$ of k defects in this segment is given by the formula

$$(1.1)\qquad P_k(h) = \frac{(ch)^k}{k!}e^{-ch} \qquad (k = 0, 1, 2, \ldots),$$

h being the length of the segment, and that the numbers of defects in disjoint segments of the ribbon are stochastically independent. We shall call the parameter c of the process the *defectiveness* of the ribbon.

Besides these assumptions characterizing the defects on each ribbon as a Poisson process we shall assume the mutual independence of any two random variables v_1 and v_2 defined by the distribution of defects on the first and on the second ribbon respectively.

§ 2. BAYES RULE AND LIKELIHOOD

Before we proceed to the comparison of two processes we may discuss some questions related to a single process.

Let us observe a homogeneous Poisson process in the interval of unit length. Let us suppose that we have observed k defects, or, in other words, that $\varkappa = k$, where $\varkappa$ denotes the random variable representing the number of defects. What is the probability that the parameter c of this process is less than a? This question cannot be answered without the knowledge of the prior distribution of the parameter, *i. e.*, without a knowledge preceding the observation and determining before the observation the sought probability. Usually, we lack this knowledge. In these cases the so called "fiducial argument" is used which dispenses with this knowledge. This argument consists in changing the original question

and in computing another probability which can be determined without the knowledge of the prior distribution. This probability is then called the *likelihood* of the inequality $c < a$.

In his papers [5], [6], [7] Steinhaus has discussed the difficulties connected with the lack of information about the prior distributions and tried to explain the role of likelihood or "fiducial probability". In connection with these considerations Sarkadi published in 1953 the paper [4] (written in Hungarian). The word "duality" in the title of his paper reminds Oderfeld's paper [1]. Now, reproducing here one of Sarkadi's results, we shall show the "fiducial argument" and the rule of dualism for the Poisson process.

The "fiducial argument" substitutes for the previous question of the probability of c being less than a, provided that in the unit length of the ribbon k defects have been observed, the following one: what is the probability of observing more than k defects in the unit of length of the ribbon if the defectiveness of the inspected ribbon is equal to a, *i. e.*, if $c = a$? Let us denote the last probability by $P(\varkappa \geqslant k | c = a)$. If, moreover, the likelihood of the inequality $c < a$ is denoted by $W(c < a | \varkappa = k)$, we shall have the following relation:

$$(2.1)\qquad W(c < a \mid \varkappa = k) = P(\varkappa \geqslant k \mid c = a).$$

Now the original question becomes meaningful if we accept any prior distribution of the parameter. Further one can ask for what prior distribution of the parameter c the posterior probability of the inequality $c < a$ is equal to the likelihood of this inequality. We shall denote the posterior probability by $P_{\mathrm{HB}}(c < a | \varkappa = k)$, the letter B reminding us that this probability is calculated according to Bayes' rule and the letter H standing for the hypothesis on the prior distribution. We also ask for what prior distribution of the parameter c the relation

$$(2.2)\qquad W(c < a \mid \varkappa = k) = P_{\mathrm{HB}}(c < a \mid \varkappa = k)$$

holds.

Now relation (2.2) does not hold for any prior distribution of the parameter in the usual sense, but, as has been observed by Sarkadi [4], relation (2.2) holds if we assume for the prior distribution of the parameter the uniform distribution on the half-line $0 < c < \infty$. The precise meaning of this statement is as follows. If we define the right side of (2.2) by the relation

$$(2.3)\qquad P_{\mathrm{HB}}(c < a \mid \varkappa = k) = \lim_{L\to\infty} P_{\mathrm{HB}}^{(L)}(c < a \mid \varkappa = k),$$

where $P_{\mathrm{HB}}^{(L)}(c < a \mid \varkappa = k)$ is the posterior probability of $c < a$ calculated according to Bayes' rule under the hypothesis that on the unit length of the ribbon k defects have been observed and that c is *a priori* uniformly distributed in the interval $0 < c < L$, then we have (2.2) for every k and a.

We call (2.2) the *rule of dualism* for the Poisson process. This rule reveals the arbitrary hypothesis concealed in the notion of likelihood which, at first sight, has nothing to do with the prior distributions.

For the sake of completeness we give here an easy proof of (2.2). Let

$$p_L(a) = \begin{cases} 1/L & \text{for} \quad 0 < a < L, \\ 0 & \text{otherwise}, \end{cases} \tag{2.4}$$

be the probability density of the prior distribution of the parameter c. By virtue of the assumption we have

$$P(\varkappa = k \mid c = a) = \frac{a^k}{k!}e^{-a}. \tag{2.5}$$

Using Bayes' rule we get

$$P_{\mathrm{HB}}^{(L)}(c < a \mid \varkappa = k) = \frac{\int\limits_0^a p_L(a)P(\varkappa = k \mid c = a)\,da}{\int\limits_0^L p_L(a)P(\varkappa = k \mid c = a)\,da} = \frac{\int\limits_0^a \frac{1}{L}\frac{a^k}{k!}e^{-a}da}{\int\limits_0^L \frac{1}{L}\frac{a^k}{k!}e^{-a}da}.$$

Obviously we have

$$\int\limits_a^b \frac{a^k}{k!}e^{-a}da = e^{-a}\left(1+a+\ldots+\frac{a^k}{k!}\right) - e^{-b}\left(1+b+\ldots+\frac{b^k}{k!}\right),$$

and therefore

$$P_{\mathrm{HB}}^{(L)}(c < a \mid \varkappa = k) = \frac{1-e^{-a}\left(1+a+\ldots+\frac{a^k}{k!}\right)}{1-e^{-L}\left(1+L+\ldots+\frac{L^k}{k!}\right)}.$$

This and (2.3) imply

$$P_{\mathrm{HB}}(c < a \mid \varkappa = k) = 1-e^{-a}\left(1+a+\ldots+\frac{a^k}{k!}\right),$$

and we immediately get from (2.1)

$$W(c < a \mid \varkappa = k) = \sum_{i=k+1}^{\infty} e^{-a}\frac{a^i}{i!} = 1-e^{-a}\left(1+a+\ldots+\frac{a^k}{k!}\right),$$

which proves (2.2).

§ 3. *RULES OF DUALISM FOR THE COMPARISON OF TWO PROCESSES*

We now turn back to the comparison of two processes. We shall denote by c_{I} and c_{II} the parameters of the first and the second process respectively in the sense of § 1. The random variables representing the numbers of defects observed during the examination on the first and the second ribbon shall be denoted by ν and μ respectively. For obvious reasons the joint probability distribution of the variables ν and μ does not depend on the absolute values of the parameters c_{I} and c_{II}, but only on the ratio $c_{\mathrm{II}}/c_{\mathrm{I}}$. Therefore it is the distribution of the ratio $c_{\mathrm{II}}/c_{\mathrm{I}}$ that really matters. Accordingly we shall denote by

$$P_{\mathrm{HB}_j}(c_{\mathrm{II}}/c_{\mathrm{I}} > a \mid \mu = m, \mu+\nu = m+n),$$

the posterior probability of $c_{\mathrm{II}}/c_{\mathrm{I}}$ being greater than a if the prior distribution of the ratio $c_{\mathrm{II}}/c_{\mathrm{I}}$ is specified by the hypothesis (H_j) and if n and m defects have been observed on the first and on the second ribbon respectively, the plan being classical. If the sequential plan of examination is adopted, then the posterior probability of the same inequality shall be denoted by

$$P_{\mathrm{HB}_j}(c_{\mathrm{II}}/c_{\mathrm{I}} > a \mid \mu = m, \nu = n).$$

Further

$$P(\mu \leqslant m \mid c_{\mathrm{II}}/c_{\mathrm{I}} = a, \mu+\nu = m+n)$$

shall denote the probability that we shall observe at most m defects on the second ribbon if the examination continues until the total $m+n$ of observed defects reaches the prescribed number N (classical plan), the ratio $c_{\mathrm{II}}/c_{\mathrm{I}}$ being supposed equal to a. The analogous probability for the sequential examination shall be denoted by

$$P(\mu \leqslant m \mid c_{\mathrm{II}}/c_{\mathrm{I}} = a, \nu = n).$$

It will be shown that

$$P(\mu \leqslant m \mid c_{\mathrm{II}}/c_{\mathrm{I}} = a, \mu+\nu = m+n) = \sum_{k=0}^{m}\binom{m+n}{k}\left(\frac{a}{1+a}\right)^k\left(\frac{1}{1+a}\right)^{m+n-k} \tag{3.1}$$

and that

$$P(\mu \leqslant m \mid c_{\mathrm{II}}/c_{\mathrm{I}} = a, \nu = n) = \sum_{k=0}^{m}\binom{n-1+k}{k}\left(\frac{a}{1+a}\right)^k\left(\frac{1}{1+a}\right)^n. \tag{3.2}$$

The formula (3.2) was first found by Romejko [3] for $a = 1$.

The "fiducial argument" which abandons the Bayes rule, substitutes for posterior probabilities the likelihood of c_{II}/c_I being greater than a, provided that n and m defects have been observed on the first and on the second ribbon respectively. The likelihood W is defined for the classical plan by

(3.3) $$W(c_{II}/c_I > a \mid \mu = m, \mu+\nu = m+n) = P(\mu \leqslant m \mid c_{II}/c_I = a, \mu+\nu = m+n),$$

and for the sequential plan by

(3.4) $$W(c_{II}/c_I > a \mid \mu = m, \nu = n) = P(\mu \leqslant m \mid c_{II}/c_I = a, \nu = n).$$

Now it turns out that the likelihood is equal to the posterior probability calculated with Bayes' rule, if a suitable hypothesis on the prior probability is adopted, as is shown by the forthcoming rules of dualism. They show that the notion of likelihood is by no means free of hypotheses on prior distributions. It is also remarkable that in the most interesting cases one must assume for the prior distribution the uniform distribution on the half-line.

We shall be concerned with the following hypotheses on the prior distribution of the ratio c_{II}/c_I:

(H_0) the ratio c_{II}/c_I has the distribution function

$$P(c_{II}/c_I < a) = \begin{cases} a/(1+a) & \text{for} \quad 0 < a < \infty, \\ 0 & \text{otherwise;} \end{cases}$$

(H_1) $\log(1+c_{II}/c_I)$ is uniformly distributed on the half-line $(0, \infty)$;

(H_2) the ratio c_{II}/c_I is uniformly distributed on the half-line $(0, \infty)$.

It must be explained what is meant by the posterior probability under the hypothesis that a certain quantity is *a priori* uniformly distributed on the half-line $(0, \infty)$. We define it as the limit of posterior probability calculated under the hypothesis that this quantity is *a priori* uniformly distributed in the finite interval $(0, L)$, with $L \to \infty$.

We call attention to the fact that only the hypothesis (H_0) is symmetric with respect to c_I and c_{II}. Under this hypothesis c_{II}/c_I and c_I/c_{II} have the same distribution. The remaining hypotheses favour great values of c_{II}/c_I.

For the classical plan of examination we have the following rules of dualism:

(3.5) $$P_{HB0}(c_{II}/c_I > a \mid \mu = m, \mu+\nu = m+n) = W(c_{II}/c_I > a \mid \mu = m, \mu+\nu = m+n+1) \quad (n \geqslant 0);$$

(3.6) $$P_{HB1}(c_{II}/c_I > a \mid \mu = m, \mu+\nu = m+n) = W(c_{II}/c_I > a \mid \mu = m, \mu+\nu = m+n) \quad (n \geqslant 1);$$

(3.7) $$P_{HB2}(c_{II}/c_I > a \mid \mu = m, \mu+\nu = m+n) = W(c_{II}/c_I > a \mid \mu = m, \mu+\nu = m+n-1) \quad (n \geqslant 2).$$

The relations

(3.8) $$P_{HB_j}(c_{II}/c_I > a \mid \mu = m, \mu+\nu = m+n) = P_{HB_j}(c_{II}/c_I > a \mid \mu = m, \nu = n) \quad (j = 0, 1, 2)$$

and

(3.9) $$P(\mu \leqslant m \mid c_{II}/c_I = a, \mu+\nu = m+n) = P(\mu \leqslant m \mid c_{II}/c_I = a, \nu = n)$$

make it possible to deduce immediately from (3.5)-(3.7) the analogous rules of dualism for the sequential plan of examination. We have namely

(3.10) $$P_{HB_j}(c_{II}/c_I > a \mid \mu = m, \nu = n) = W(c_{II}/c_I > a \mid \mu = m, \nu = n+1-j) \quad (n \geqslant j;\ j = 0, 1, 2).$$

Special attention is to be paid to the rules (3.6) and (3.10) with $j = 1$, as they reveal the hypotheses concealed in the notion of likelihood while the others show only the close connection of the notions of likelihood and prior probability.

We defer the proofs of the above rules to the forthcoming sections.

§ 4. RELATION TO THE PROBLEM OF ESTIMATING FRACTION DEFECTIVE

In this section we propose to show that the comparison of two Poisson processes may be reduced to the investigation of the fraction of, say, red-coloured balls in an urn. Namely at every instant t we can ask for the probability p of the next defect to appear on the first ribbon. Let us suppose that both ribbons spoken of in the introduction run with unit speed. Further, if τ_I is the time of waiting since the instant t for the appearence of the first defect on the first ribbon and τ_{II} is the time of waiting since the instant t for the appearence of the first defect on the second ribbon, then, by our assumptions, τ_I and τ_{II} are independent random variables with distribution functions

$$P(\tau_i < t) = 1 - e^{-c_i t} \quad (i = \mathrm{I}, \mathrm{II}).$$

We observe the next defect on the first ribbon if $\tau_{\rm I} < \tau_{\rm II}$, and on the second ribbon if $\tau_{\rm II} < \tau_{\rm I}$. Now it is easily verified that

$$p = P(\tau_{\rm I} < \tau_{\rm II}) = c_{\rm I}/(c_{\rm I}+c_{\rm II}) = 1/(1+c_{\rm II}/c_{\rm I}), \tag{4.1}$$

$$q = P(\tau_{\rm II} < \tau_{\rm I}) = c_{\rm II}/(c_{\rm I}+c_{\rm II}) = (c_{\rm II}/c_{\rm I})/(1+c_{\rm II}/c_{\rm I}). \tag{4.2}$$

The equality $\tau_{\rm I} = \tau_{\rm II}$ has probability zero and may be disregarded.

It also turns out that observing defects on two ribbons running side by side and characterized by the defectivenesses $c_{\rm I}$ and $c_{\rm II}$ amounts to the same as drawing balls from an urn containing the fractions p and $q = 1-p$ of red and black balls respectively. The role of the ratio $c_{\rm II}/c_{\rm I}$ is now played by the ratio q/p of fractions of balls. The present problem differs from those usually considered in the statistical quality control by the fact that we are now interested in the ratio $r = q/p$ and not in the fraction q; in the usual model black balls are called bad and q is called the fraction defective. Nevertheless the relation $r = q/(1-q)$ makes it possible to pass immediately from r to q and conversely.

§ 5. TRANSLATION OF THE ORIGINAL PROBLEM INTO THE LANGUAGE OF DRAWING BALLS

Owing to (4.1) and (4.2) we now express our original problem in terms of drawing balls.

The classical plan of examination means now that a fixed number N of balls are drawn, while the sequential plan of examination means that the balls are drawn one after another until the total of red balls drawn reaches the prescribed number n. The relations $c_{\rm II}/c_{\rm I} > \alpha$ and $q > \beta$, where $\beta = \alpha/(1+\alpha)$, are now equivalent. To the hypotheses (H_i) on the prior distribution of the ratio $c_{\rm II}/c_{\rm I}$ correspond the following hypotheses on the prior distribution of q:

(H_0') the fraction q is uniformly distributed in the interval $(0,1)$;

(H_1') $\log[1/(1-q)]$ is uniformly distributed on the half-line $(0, \infty)$;

(H_2') the ratio $q/(1-q)$ is uniformly distributed on the half-line $(0, \infty)$.

Note that (H_0') is the only hypothesis symmetric in q and $1-q$. Under this hypothesis q and $1-q$ have the same distribution. The other two hypotheses favour great values of q and therefore seem too pessimistic from the point of view of statistical quality control.

Without danger of misunderstanding we may retain the notation μ and ν for the numbers of black and red balls drawn from the urn during the examination. Now with easily comprehensible changes of notation we obtain the following equivalents for the relations (3.1)-(3.10):

$$P(\mu \leqslant m \mid q = \beta, \mu+\nu = m+n) = \sum_{k=0}^{m} \binom{m+n}{k} \beta^k (1-\beta)^{m+n-k}; \tag{5.1}$$

$$P(\mu \leqslant m \mid q = \beta, \nu = n) = \sum_{k=0}^{m} \binom{n-1+k}{k} \beta^k (1-\beta)^n; \tag{5.2}$$

$$W(q > \beta \mid \mu = m, \mu+\nu = m+n) = P(\mu \leqslant m \mid q = \beta, \mu+\nu = m+n); \tag{5.3}$$

$$W(q > \beta \mid \mu = m, \nu = n) = P(\mu \leqslant m \mid q = \beta, \nu = n); \tag{5.4}$$

$$P_{\mathrm{HB0}'}(q > \beta \mid \mu = m, \mu+\nu = m+n) = W(q > \beta \mid \mu = m, \mu+\nu = m+n+1) \quad (n \geqslant 0); \tag{5.5}$$

$$P_{\mathrm{HB1}'}(q > \beta \mid \mu = m, \mu+\nu = m+n) = W(q > \beta \mid \mu = m, \mu+\nu = m+n) \quad (n \geqslant 1); \tag{5.6}$$

$$P_{\mathrm{HB2}'}(q > \beta \mid \mu = m, \mu+\nu = m+n) = W(q > \beta \mid \mu = m, \mu+\nu = m+n-1) \quad (n \geqslant 2); \tag{5.7}$$

$$P_{\mathrm{HB}j'}(q > \beta \mid \mu = m, \mu+\nu = m+n) = P_{\mathrm{HB}j'}(q > \beta \mid \mu = m, \nu = n) \quad (j = 0, 1, 2); \tag{5.8}$$

$$P(\mu \leqslant m \mid q = \beta, \mu+\nu = m+n) = P(\mu \leqslant m \mid q = \beta, \nu = n); \tag{5.9}$$

$$P_{\mathrm{HB}j'}(q > \beta \mid \mu = m, \nu = n) = W(q > \beta \mid \mu = m, \nu = n+1-j) \quad (n \geqslant j;\ j = 0, 1, 2). \tag{5.10}$$

The rule (5.5) was first discovered by Oderfeld [1]. Sarkadi [4] has remarked that it is a special case of the rules of dualism which correspond to the hypothesis of q having *a priori* the beta distribution.

§ 6. PROOFS

To render all our statements fully proved it remains to establish the relations (5.1), (5.2) and (5.5)-(5.9).

We have

$$P(\mu = m \mid q = \beta, \mu+\nu = m+n) = \binom{m+n}{m} \beta^m (1-\beta)^n \tag{6.1}$$

and

$$P(\mu = m \mid q = \beta, \nu = n) = \binom{m+n-1}{m} \beta^m (1-\beta)^n. \tag{6.2}$$

The first equality is obvious as it states that the number μ of black balls in the sample of size $N=m+n$ drawn at random from the urn containing the fraction β of black balls has the binomial distribution. As to (6.2) let us remark that if we draw balls until the n-th red ball is drawn, we obtain m black balls if and only if in the first $m+n-1$ drawing m black and $n-1$ red balls are drawn and in the last, *i. e.*, the $(m+n)$-th drawing there, appears a red ball. The first of these events has the probability $(m+n-1)\beta^m(1-\beta)^{n-1}$, while $1-\beta$ is the probability of the second one. The right side of (6.2) is already the product of these two probabilities.

Relations (5.1) and (5.2) follow immediately from (6.1) and (6.2).

In the sequel we shall use the identities

$$(6.3)\qquad \frac{1}{B(m+1,n+1)}\int_a^b w^m(1-w)^n dw = \sum_{k=0}^{m}\binom{n+m+1}{k}a^k(1-a)^{n+m+1-k}-\sum_{k=0}^{m}\binom{n+m+1}{k}b^k(1-b)^{n+m+1-k}$$

and

$$(6.4)\qquad \frac{1}{B(m+1,n+1)}\int_a^b w^m(1-w)^n dw = \sum_{k=0}^{m}\binom{n+k}{k}a^k(1-a)^{n+1}-\sum_{k=0}^{m}\binom{n+k}{k}b^k(1-b)^{n+1}.$$

These identities can easily be verified by differentiation.

LEMMA. *If for a natural s*

$$(6.5)\qquad p_L(\beta)=\begin{cases}\dfrac{1}{L}\,\dfrac{1}{(1-\beta)^s}, & 0<\beta<K(L),\\ 0 & \textit{otherwise},\end{cases}$$

is the probability-density of the prior distribution of q, and if

$$P_{\mathrm{HB}}^{(L)}(q>\beta\mid\mu=m,\mu+\nu=m+n)\quad \textit{and}\quad P_{\mathrm{HB}}^{(L)}(q>\beta\mid\mu=m,\nu=n)$$

denote the posterior probabilities of q being greater than β for classical and sequential plan, then we have for $n\geqslant s$

$$(6.6)\qquad \lim_{L\to\infty}P_{\mathrm{HB}}^{(L)}(q>\beta\mid\mu=m,\mu+\nu=m+n)$$

$$=\lim_{L\to\infty}P_{\mathrm{HB}}^{(L)}(q>\beta\mid\mu=m,\nu=n)=\frac{1}{B(m+1,n+1-s)}\int_\beta^1\beta^m(1-\beta)^{n-s}d\beta.$$

Proof of the lemma. In view of (6.5) we have

$$(6.7)\qquad \lim_{L\to\infty}K(L)=1.$$

Using Bayes rule we get for the classical plan

$$(6.8)\qquad P_{\mathrm{HB}}^{(L)}(q>\beta\mid\mu=m,\mu+\nu=m+n)$$

$$=\frac{\int_\beta^{K(L)}P_L(a)P(\mu=m\mid q=a,\mu+\nu=m+n)da}{\int_0^{K(L)}p_L(a)P(\mu=m\mid q=a,\mu+\nu=m+n)da}$$

$$=\frac{\int_\beta^{K(L)}\frac{1}{L}\frac{1}{(1-a)^s}\binom{m+n}{m}a^m(1-a)^n da}{\int_0^{K(L)}\frac{1}{L}\frac{1}{(1-a)^s}\binom{m+n}{m}a^m(1-a)^n da}=\frac{\int_\beta^{K(L)}a^m(1-a)^{n-s}da}{\int_0^{K(L)}a^m(1-a)^{n-s}da}.$$

In the same way, the only change consisting in replacing $\binom{m+n}{m}$ by $\binom{m+n-1}{m}$, we obtain for the sequential plan

$$(6.9)\qquad P_{\mathrm{HB}}^{(L)}(q>\beta\mid\mu=m,\nu=n)=\frac{\int_\beta^{K(L)}a^m(1-a)^{n-s}da}{\int_0^{K(L)}a^m(1-a)^{n-s}da}.$$

In view of (6.7) we have for $n\geqslant s$

$$(6.10)\qquad \lim_{L\to\infty}\frac{\int_\beta^{K(L)}a^m(1-a)^{n-s}da}{\int_0^{K(L)}a^m(1-a)^{n-s}da}=\frac{\int_\beta^{1}a^m(1-a)^{n-s}da}{\int_0^{1}a^m(1-a)^{n-s}da}.$$

Relations (6.8)-(6.10) prove (6.6).

Proof of (5.7). If the ratio $q/(1-q)$ is *a priori* uniformly distributed in the interval $(0,L)$, then we have

$$P\left(\frac{q}{1-q}<\beta\right)=P\left(q<\frac{\beta}{1+\beta}\right)=\frac{\beta}{L}\quad\text{for}\quad 0<\beta<L,$$

or, substituting a for $\beta/(1+\beta)$,

$$P(q<a)=\frac{1}{L}\,\frac{a}{1-a}\quad\text{for}\quad 0<a<\frac{L}{1+L}.$$

The probability density of this prior distribution is

$$p_L(a) = \frac{1}{L}\frac{1}{(1-a)^2} \quad \text{for} \quad 0 < a < \frac{L}{1+L}.$$

By virtue of the lemma we have for $n \geqslant 2$

(6.11) $P_{\mathrm{HB2'}}(q > \beta \mid \mu = m,\ \mu+\nu = m+n)$

$$= P_{\mathrm{HB2'}}(q > \beta \mid \mu = m,\ \nu = n) = \frac{1}{B(m+1,\ n-1)}\int_\beta^1 a^m(1-a)^{n-2}da.$$

Note that this proves (5.8) for $j = 2$. Further, developping the right side of this equality according to (6.3), we get

$$P_{\mathrm{HB2'}}(q > \beta \mid \mu = m,\ \mu+\nu = m+n) = \sum_{k=0}^{m}\binom{m+n-1}{k}\beta^k(1-\beta)^{m+n-1-k},$$

and in view of (6.1) we recognize in the last sum the probability $P(\mu \leqslant m | q = \beta,\ \mu+\nu = m+n-1)$. This and (5.3) prove (5.7).

The proof of (5.6) is analogous to the preceding one. If $\log(1/(1-q))$ is *a priori* uniformly distributed in the interval $(0, L)$, then we have

$$P\left(\log\frac{1}{1-q} < \beta\right) = P\left(\frac{1}{1-q} < e^\beta\right) = P(q < 1-e^{-\beta}) = \frac{\beta}{L} \quad \text{for} \quad 0 < \beta < L,$$

or, substituting a for $1-e^{-\beta}$,

$$P(q < a) = \frac{1}{L}\log\frac{1}{1-a} \quad \text{for} \quad 0 < a < 1-e^{-L}.$$

The probability density of this prior distribution is

$$p_L(a) = \frac{1}{L}\frac{1}{1-a} \quad \text{for} \quad 0 < a < 1-e^{-L}.$$

By virtue of the lemma we have for $n \geqslant 1$

(6.12) $P_{\mathrm{HB1'}}(q > \beta \mid \mu = m,\ \mu+\nu = m+n)$

$$= P_{\mathrm{HB1'}}(q > \beta \mid \mu = m,\ \nu = n) = \frac{1}{B(m+1,\ n)}\int_\beta^1 a^m(1-a)^{n-1}da.$$

This proves (5.8) for $j = 1$. Further, if we develop the last integral according to (6.3) and take into account (6.1) and (5.3), we obtain (5.6).

The proof of (5.5) and of (5.8) for $j = 0$ proceeds in the same manner, with the only change that we deal here with the prior distribution in the usual sense so that the passage to the limit is spared.

Finally, by virtue of (6.3) and (6.4), we can write

$$\frac{1}{B(m+1,\ n)}\int_\beta^1 w^m(1-w)^{n-1}dw = \sum_{k=0}^{m}\binom{n+m}{k}\beta^k(1-\beta)^{n+m-k}$$

$$= \sum_{k=0}^{m}\binom{n-1+k}{k}\beta^k(1-\beta)^n,$$

which, in view of (5.1) and (5.2), proves (5.9). This relation enables us to use Pearson's tables [2] for the calculation of probabilities and likelihoods.

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Reçu par la Rédaction le 22. 10. 1956