Nous avons ainsi

$$\tfrac{1}{2}k+\int_{S_n}\|x-\lambda_n\|\,dP \leqslant \|b-\lambda_n\|+\int_{S-S_n}\|x\|\,dP-n\cdot \mathrm{mes}(S-S_n);$$

et en conséquence [4])

$$\tfrac{1}{2}k+\int_{S}\|x-\lambda_n\|\,dP \leqslant \|b-\lambda_n\|+2\int_{S-S_n}\|x\|\,dP.$$

Comme $E\|x\| < +\infty$ pour n assez grand, nous obtenons

$$E\|x-\lambda_n\| < \|b-\lambda_n\|,$$

ce qui montre que b n'est pas une moyenne au sens de Doss.

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[4]) $n\cdot \mathrm{mes}(S-S_n) = \|\lambda_n\|\cdot \mathrm{Pr}(S-S_n)$.

REMARKS ON THE DOSS INTEGRAL

BY

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I. In this note $\mathfrak{X}$ will denote a metric space, I — the interval $0 \leqslant t \leqslant 1$ and $\mathfrak{X}^I$ — the set of all $\mathfrak{X}$-valued functions defined on I.

A function f ($f\in\mathfrak{X}^I$) is called *measurable* if for every open subset U ($U\subset\mathfrak{X}$) $f^{-1}(U)$ is a Lebesgue measurable subset of I.

A measurable function f ($f\in\mathfrak{X}^I$) is called *Doss integrable* (see [1]) if there exists a *unique* element $a_f\in\mathfrak{X}$ such that for every $z\in\mathfrak{X}$ the inequality

$$\varrho(a_f, z) \leqslant \int_0^1 \varrho(f(t), z)\,dt$$

holds (ϱ denotes the distance in $\mathfrak{X}$). The element a_f is called the *Doss integral* of the function f. We use the notation

$$a_f = \int_0^1 f(t)\,dt.$$

Suppose the following intepretation: t is a random parameter, and consequently a measurable function f is a random variable. The Doss integral a_f is the expectation of a random variable f. The purpose of this note is to prove that, with some natural assumptions concerning a space $\mathfrak{X}$, if for every finitely-valued random variable there exists an expectation, then $\mathfrak{X}$ is a normed linear space.

Finally we remark that the result of the present note is also true if the interval I with the Lebesgue measure is replaced by an arbitrary measure space with a non-atomic probability measure.

II. Suppose that $\mathfrak{X}$ is an abelian metric group. Let $+$ denote the group-addition and Θ the zero element. It is well known that for every metric group there exists an invariant distance, *i. e.*, a distance satisfying for all $x, y, z\in\mathfrak{X}$ the condition $\varrho(x+z, y+z) = \varrho(x, y)$ (see [2]).

$\mathfrak{A}$ will denote the set of all measurable functions belonging to $\mathfrak{X}^I$ which only take a finite number of values.

THEOREM. *If $\mathfrak{X}$ is an abelian metric group with an invariant distance and if every function belonging to $\mathfrak{A}$ is Doss integrable, then $\mathfrak{X}$ is a normed linear space and the Doss integral over $\mathfrak{A}$ is equal to the usual integral, i. e.,*

$$\int_0^1 f(t)\,dt = \sum_{j=1}^{n} m(I_j)x_j, \tag{1}$$

where $f(t) = x_j \in \mathfrak{X}$ for $t \in I_j$ $(j = 1, 2, \ldots, n)$, $I = \bigcup_{j=1}^{n} I_j$ and $I_j \cap I_k = 0$ for $j \neq k$ ($m(A)$ denotes the Lebesgue measure of the set A).

Remarks. (a) The assumption of the invariance of distance ϱ is essential.

Indeed, let $\mathfrak{X}$ be the multiplicative group of all positive real numbers with the distance $\varrho(x, y) = |x-y|$. Assume $x_j > 0$, $\lambda_j \geqslant 0$ $(j = 1, 2, \ldots, n)$ and $\lambda_1+\lambda_2+\ldots+\lambda_n = 1$. It is easy to prove that $a = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n$ is the unique positive real number satisfying for all $z > 0$ the inequality

$$|a-z| \leqslant \sum_{j=1}^{n} \lambda_j |x_j - z|.$$

Thus all functions belonging to $\mathfrak{A}$ are Doss integrable. It is easy to see that in this example the Doss integral is non-additive (in the sense of group multiplication). Since from (1) it follows that the Doss integral is additive, then for this example the assertion of the theorem is false.

(b) The following example shows that a space $\mathfrak{X}$ satisfying the assumptions of the theorem can be non-complete, *i. e.*, can be not a Banach space.

Let $\mathfrak{X}$ be the space of all sequences of real numbers $x = (x_1, x_2, \ldots)$ vanishing for sufficiently large indices. The addition and the scalar-multiplication are defined in the following way:

$$x+y = (x_1+y_1, x_2+y_2, \ldots), \qquad \lambda x = (\lambda x_1, \lambda x_2, \ldots).$$

The distance is defined by the formula

$$\varrho(x, y) = \sqrt{\sum_{j=1}^{\infty} (x_j - y_j)^2}.$$

Consequently, the vector space $\mathfrak{X}$ is non-complete. It is also easy to prove that every function belonging to $\mathfrak{A}$ is Doss integrable.

III. In this part suppose the assumptions of the theorem to be satisfied.

Let

$$\|x\| = \varrho(x, \Theta) \quad \text{for} \quad x \in \mathfrak{X}. \tag{2}$$

Then, obviously

(*) *An element a_f $(a_f \in \mathfrak{X})$ is the Doss integral of a function f $(f \in \mathfrak{A})$ if and only if for each $z \in \mathfrak{X}$ the inequality*

$$\|a_f - z\| \leqslant \int_0^1 \|f(t) - z\|\,dt$$

holds.

To prove the theorem it is sufficient

(α) to define for every $x \in \mathfrak{X}$ and for every real number λ the product λx satisfying the following conditions:

(i) $$(\lambda_1+\lambda_2)x = \lambda_1 x + \lambda_2 x,$$

(ii) $$\lambda(x_1+x_2) = \lambda x_1 + \lambda x_2,$$

(iii) $$\lambda_1(\lambda_2 x) = (\lambda_1\lambda_2)x,$$

(iv) $$1 \cdot x = x,$$

(v) $$\|\lambda x\| = |\lambda| \cdot \|x\|.$$

The second part of theorem is the immediate consequence of (α) and the definition of the Doss integral. Indeed, from (α) it follows that for each $z \in \mathfrak{X}$ the following inequality holds:

$$\Big\| \sum_{j=1}^{n} m(I_j)x_j - z \Big\| \leqslant \sum_{j=1}^{n} m(I_j) \big\| x_j - z \big\| = \int_0^1 \big\| f(t) - z \big\|\,dt.$$

Hence, according to (*), we obtain equality (1).

Before the proof of condition (α) we shall prove some elementary lemmas. The definition of the product λx is contained in formulae (3), (4) and (11).

Let $x \in \mathfrak{X}$ and $0 \leqslant \nu \leqslant 1$. Let

$$f(\nu, x, t) = \begin{cases} x & \text{for} \quad 0 \leqslant t \leqslant \nu, \\ \Theta & \text{for} \quad \nu < t \leqslant 1. \end{cases} \tag{3}$$

It is easy to see that, for fixed x and ν, $f(\nu, x, t) \in \mathfrak{A}$. By $D(\nu, x)$ we shall denote by $D(\nu, x)$ the Doss integral of the function $f(\nu, x, t)$:

$$D(\nu, x) = \int_0^1 f(\nu, x, t)\,dt. \tag{4}$$

LEMMA 1. *Let $x \in \mathfrak{X}$. Then the equality $2x = \Theta$ implies the equality $x = \Theta$.*

Proof. In view of (3) and (4) the inequality (*) will take for $\nu = \frac{1}{2}$ the form

$$\|D(\tfrac{1}{2}, x) - z\| \leqslant \int_0^1 \|f(\tfrac{1}{2}, x, t) - z\|\,dt \leqslant \tfrac{1}{2}\|x - z\| + \tfrac{1}{2}\|z\| \quad (z \in \mathfrak{X}). \tag{5}$$

This implies

$$\|D(\tfrac{1}{2}, x)+x-z\| \leqslant \tfrac{1}{2}\|2x-z\|+\tfrac{1}{2}\|x-z\| \qquad (z \in \mathcal{X}).$$

Since $2x = \Theta$, the last inequality implies

$$\|D(\tfrac{1}{2}, x)+x-z\| \leqslant \tfrac{1}{2}\|x-z\|+\tfrac{1}{2}\|z\| \qquad (z \in \mathcal{X}).$$

Consequently, in view of (5), we have $D(\frac{1}{2}, x)+x = D(\frac{1}{2}, x)$. Then we find that $x = \Theta$.

LEMMA 2. *For each* $x \in \mathcal{X}$ *and* $0 \leqslant \nu \leqslant 1$ *the equality* $D(\nu, x)+D(1-\nu, x) = x$ *is true.*

Proof. From (*), (3) and (4) it follows that the following inequality holds:

$$\|D(\nu, x)-z\| \leqslant \nu\|x-z\|+(1-\nu)\|z\| \qquad (z \in \mathcal{X}).$$

Then, by replacing z by $x-z$, we get

$$\|x-D(\nu, x)-z\| \leqslant (1-\nu)\|x-z\|+\nu\|z\| \qquad (z \in \mathcal{X}).$$

Consequently,

$$x-D(\nu, x) = \int_0^1 f(1-\nu, x, t)\,dt = D(1-\nu, x).$$

The lemma is thus proved.

LEMMA 3. *For each* $x \in \mathcal{X}$ *the equality* $D(\frac{1}{2}, 2x) = x$ *is true.*

Proof. According to lemma 2 we obtain $2D(\frac{1}{2}, 2x) = 2x$. Consequently, $2\{D(\frac{1}{2}, 2x)-x\} = \Theta$. Then, in view of lemma 1, $D(\frac{1}{2}, 2x) = x$.

LEMMA 4. *For each* $x \in \mathcal{X}$ *the equality*

$$\|2x\| = 2\|x\| \tag{6}$$

is true.

Proof. From (3), (4) and (*) (for $z = \Theta$) it follows that the following inequality holds:

$$\|D(\tfrac{1}{2}, 2x)\| \leqslant \tfrac{1}{2}\|2x\|.$$

Then, in view of lemma 3, $2\|x\| \leqslant \|2x\|$. Hence, taking into account the triangle inequality $\|2x\| \leqslant 2\|x\|$, we obtain equality (6).

LEMMA 5. *If* $0 \leqslant \nu_1, \nu_2 \leqslant 1$ *and* $\nu_1+\nu_2 \leqslant 1$, *then for each* $x \in \mathcal{X}$ *the equality*

$$D(\nu_1, x)+D(\nu_2, x) = D(\nu_1+\nu_2, x)$$

is true.

Proof. From (*), (3) and (4) it follows that for each z the following inequalities hold:

$$\|D(\nu_1, x)-z\| \leqslant \nu_1\|x-z\|+(1-\nu_1)\|z\|, \tag{7}$$

$$\|D(\nu_2, x)-z\| \leqslant \nu_2\|x-z\|+(1-\nu_2)\|z\|, \tag{8}$$

$$\|D(\nu_1+\nu_2, x)-z\| \leqslant (\nu_1+\nu_2)\|x-z\|+(1-\nu_1-\nu_2)\|z\|. \tag{9}$$

Inequality (7), in which z is replaced by $z-D(\nu_2, x)$, and inequality (8) imply

$$\|D(\nu_1, x)+D(\nu_2, x)-z\| \leqslant \nu_1\nu_2\|2x-z\|+ \\ +(\nu_1+\nu_2-2\nu_1\nu_2)\|x-z\|+(1+\nu_1\nu_2-\nu_1-\nu_2)\|z\|. \tag{10}$$

From the inequality $\|2(x-z)\| \leqslant \|2x-z\|+\|z\|$ and from Lemma 4 we obtain

$$0 \leqslant \|2x-z\|+\|z\|-2\|x-z\|.$$

Thus in view of (9) the following inequality is satisfied:

$$\begin{aligned}\|D(\nu_1+\nu_2, x)-z\| &\leqslant (\nu_1+\nu_2)\|x-z\|+(1-\nu_1-\nu_2)\|z\|+\nu_1\nu_2(\|2x-z\|+\|z\|-2\|x-z\|)\\ &= \nu_1\nu_2\|2x-z\|+(\nu_1+\nu_2-2\nu_1\nu_2)\|x-z\|+(1+\nu_1\nu_2-\nu_1-\nu_2)\|z\|.\end{aligned}$$

Hence, according to (10),

$$\|D(\nu_1+\nu_2, x)-z\| \leqslant \int_0^1 \|f_0(t)-z\|\,dt,$$

$$\|D(\nu_1, x)+D(\nu_2, x)-z\| \leqslant \int_0^1 \|f_0(t)-z\|\,dt,$$

where the function f_0 ($f_0 \in \mathfrak{A}$) is defined by the formula

$$f_0(t) = \begin{cases} 2x & \text{for} \quad 0 \leqslant t \leqslant \nu_1\nu_2, \\ x & \text{for} \quad \nu_1\nu_2 < t \leqslant \nu_1+\nu_2-\nu_1\nu_2, \\ \Theta & \text{for} \quad \nu_1+\nu_2-\nu_1\nu_2 < t \leqslant 1. \end{cases}$$

From the last inequalities and from (*) we see that the equality $D(\nu_1, x)+D(\nu_2, x) = D(\nu_1+\nu_2, x)$ is true. The lemma is thus proved.

LEMMA 6. *If* $0 \leqslant \nu_1, \nu_2 \leqslant 1$ *and* $\nu_1+\nu_2 > 1$, *then for each* $x \in \mathcal{X}$ *the equality*

$$D(\nu_1, x)+D(\nu_2, x) = D(\nu_1+\nu_2-1, x)+x$$

is true.

Proof. According to lemma 5

$$D(\nu_2, x) = D(1-\nu_1, x)+D(\nu_1+\nu_2-1, x).$$

This implies

$$D(\nu_1, x)+D(\nu_2, x) = D(\nu_1+\nu_2-1, x)+D(\nu_1, x)+D(1-\nu_1, x).$$

Hence, in view of lemma 2, the assertion of the lemma is obtained.

LEMMA 7. *For each $x \in \mathfrak{X}$ and $0 \leqslant \nu_1, \nu_2 \leqslant 1$ the equality*

$$D(\nu_1, D(\nu_2, x)) = D(\nu_1\nu_2, x)$$

is true.

Proof. From (7) by replacing x by $D(\nu_2, x)$ we get the inequality

$$\|D(\nu_1, D(\nu_2, x))-z\| \leqslant \nu_1\|D(\nu_2, x)-z\|+(1-\nu_1)\|z\| \quad (z \in \mathfrak{X}),$$

Hence, according to (8),

$$\|D(\nu_1, D(\nu_2, x))-z\| \leqslant \nu_1(\nu_2\|x-z\|+(1-\nu_2)\|z\|)+(1-\nu_1)\|z\|$$
$$= \nu_1\nu_2\|x-z\|+(1-\nu_1\nu_2)\|z\| \quad (z \in \mathfrak{X}).$$

Then, in view of (*) and (3), $D(\nu_1, D(\nu_2, x))$ is the Doss integral of the function $f(\nu_1\nu_2, x, t)$. Hence, according to (4), the assertion of the lemma is obtained.

Proof of theorem. The product λx, where $x \in \mathfrak{X}$ and λ is a real number, is defined by the formula

$$\lambda x = [\lambda]x+D(\lambda-[\lambda], x) \tag{11}$$

($[\lambda]$ denotes the greatest integer $\leqslant \lambda$).

From lemmas 5 and 6 it follows that for each λ_1, λ_2 and $x \in \mathfrak{X}$ the following equality holds:

$$D(\lambda_1-[\lambda_1], x)+D(\lambda_2-[\lambda_2], x)$$
$$= D(\lambda_1+\lambda_2-[\lambda_1+\lambda_2], x)+([\lambda_1+\lambda_2]-[\lambda_1]-[\lambda_2])x.$$

Hence, according to (11),

$$(\lambda_1+\lambda_2)x = [\lambda_1+\lambda_2]x+D(\lambda_1+\lambda_2-[\lambda_1+\lambda_2], x)$$
$$= [\lambda_1]x+D(\lambda_1-[\lambda_1], x)+[\lambda_2]x+D(\lambda_2-[\lambda_2], x) = \lambda_1 x+\lambda_2 x.$$

The relation (i) is thus proved.

For each $x, y \in \mathfrak{X}$, in view of relation (i), we have

$$2^n\left\{\frac{m}{2^n}(x+y)\right\} = m(x+y) \quad (n = 0, 1, \ldots; m = 0, \pm 1, \ldots).$$

Hence

$$m(x+y) = mx+my = 2^n\left\{\frac{m}{2^n}x+\frac{m}{2^n}y\right\} \quad (n = 0, 1, \ldots; m = 0, \pm 1, \ldots).$$

This implies

$$2^n\left\{\frac{m}{2^n}(x+y)-\left(\frac{m}{2^n}x+\frac{m}{2^n}y\right)\right\} = \Theta \quad (n = 0, 1, \ldots; m = 0, \pm 1, \ldots).$$

Consequently, in view of lemma 1, we have for each $x, y \in \mathfrak{X}$

$$\frac{m}{2^n}(x+y) = \frac{m}{2^n}x+\frac{m}{2^n}y \quad (n = 0, 1, \ldots; m = 0, \pm 1, \ldots). \tag{12}$$

From (*) (for $z = \Theta$), (3) and (4) it follows that for each $x \in \mathfrak{X}$ the following inequality holds:

$$\|D(\lambda-[\lambda], x)\| \leqslant (\lambda-[\lambda])\|x\|. \tag{13}$$

Definition (11) implies

$$\|\lambda x\| \leqslant \|[\lambda]x\|+\|D(\lambda-[\lambda], x)\|.$$

If $\lambda \geqslant 0$, then the last inequality and (13) imply

$$\|\lambda x\| \leqslant [\lambda]\|x\|+(\lambda-[\lambda])\|x\| = \lambda\|x\|.$$

If $\lambda < 0$, then, according to (i), $(-\lambda)x = -\lambda x$, and consequently

$$\|\lambda x\| = \|(-\lambda)x\| \leqslant (-\lambda)\|x\|.$$

We see that for each $x \in \mathfrak{X}$ and for each real number λ the inequality

$$\|\lambda x\| \leqslant |\lambda|\cdot\|x\| \tag{14}$$

is true. If $\lambda_n \to 0$, then the last inequality implies $\lambda_n x \to \Theta$. Hence, in view of (i), we obtain the following implication: if $\lambda_n \to \lambda$, then for $x \in \mathfrak{X}$, $\lambda_n x \to \lambda x$. Hence, according to (12), it follows that the relation (ii) holds for each real number λ and for each $x, y \in \mathfrak{X}$.

The relation (i) and (ii) imply immediately the following equalities:

$$\lambda_1(\lambda_2 x) = \lambda_1([\lambda_2]x)+[\lambda_1]\{(\lambda_2-[\lambda_2])x\}+(\lambda_1-[\lambda_1])\{(\lambda_2-[\lambda_2])x\}, \tag{15}$$

$$\lambda_1([\lambda_2]x) = (\lambda_1[\lambda_2])x, \tag{16}$$

$$[\lambda_1]\{(\lambda_2-[\lambda_2])x\} = \{[\lambda_1](\lambda_2-[\lambda_2])\}x. \tag{17}$$

From definition (11) and from lemma 7 we obtain

$$(\lambda_1-[\lambda_1])\{(\lambda_2-[\lambda_2])x\} = \{(\lambda_1-[\lambda_1])(\lambda_2-[\lambda_2])\}x. \tag{18}$$

The equalities (i), (15), (16), (17) and (18) imply the following equality:

$$\lambda_1(\lambda_2 x) = (\lambda_1[\lambda_2])x+\{[\lambda_1](\lambda_2-[\lambda_2])\}x+\{(\lambda_1-[\lambda_1])(\lambda_2-[\lambda_2])\}x = (\lambda_1\lambda_2)x.$$

Relation (iii) is thus proved.

Inequality (13) implies $D(0, x) = \Theta$. Then, in view of (11), relation (iv) holds.

From inequality (14) and from relation (iii) we obtain for $\lambda \neq 0$ and $x \in \mathfrak{X}$:

$$\|x\| = \left\| \frac{1}{\lambda} (\lambda x) \right\| \leqslant \frac{1}{|\lambda|} \|\lambda x\|.$$

Hence

$$|\lambda| \|x\| \leqslant \|\lambda x\|.$$

Then, according to (14), relation (v) holds. The theorem is thus proved.

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ON THE COMPARISON OF TWO PRODUCTION PROCESSES AND THE RULE OF DUALISM*

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INTRODUCTION

A silk ribbon produced by an automaton is an example of a continuous production process. An observer estimates the quality of the product by counting the defects, say, in a given segment of the product. In this example the role of defects may be played by stains or holes. From the number of defects observed the inspector estimates the defectiveness of the product and expresses it by the number of defects per meter of ribbon.

It may happen that not the estimation of defectiveness but the comparison of two production processes is the purpose of examination. In such a case the above example should be replaced by another, namely by two ribbons running parallel with the same speed. The comparison is a statement that the defectiveness of the second ribbon is at least a times greater than the defectiveness of the first ribbon; it is known that such statements can be deduced from observations with a certain "probability". The observation proceeds according to a certain plan. We shall be concerned with two such plans. The first of them, called *classical*, consists in observing the ribbons until the total of defects in both ribbons reaches a prescribed number N. If n and m are the numbers of defects on the first and second ribbon respectively, we have $m+n = N$. The second plan, called *sequential*, consists in observing the ribbons until on the first ribbon the n-th defect appears, where n is prescribed. Once the plan is chosen and the observation taken we compute by a suitable formula the "probability" P of the statement formulated above. This probability depends on three numbers, a, m and n. Our aim is to discuss the methods of defining the "probability" P.

It is worth while to answer the question why we do not estimate the defectiveness of each process separately. We have here an analogy

* Another version of this paper appeared in Polish, see [8].