ON A PROBLEM OF BANACH

BY

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Banach [1] has asked for a characterization of those metric spaces which can be mapped in a continuous biunique fashion onto some compact metric space, and in particular has inquired whether the Banach space (c_0) can be mapped in this way.

Relevant results were obtained by Sikorski [7] and Katetov [3], but the case of (c_0) was not settled.

We remark here that *for* (c_0) *the answer is affirmative*, reasoning as follows: by a recent result of Kadec [2], (c_0) is homeomorphic with (l^1); by Mazur's theorem [6], (l^1) is homeomorphic with (l^2); by a theorem of the author [4], (l^2) is homeomorphic with its unit cell $C_n = \{x: \|x\| \leqslant 1\}$ (in the norm topology). Now the natural map of C_n onto C_w, the unit cell in the weak topology, is continuous, and C_w is known [5] to be homeomorphic with the Hilbert parallelotope P. Thus (c_0) admits a continuous biunique map onto P, which is a compact metric space, and the proof is complete.

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ON THE NOTION OF UNIFORM CONVERGENCE WITH RESPECT TO A FUNDAMENTAL SET OF FUNCTIONALS, AND ITS APPLICATION

BY

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When formulating the mean-value theorem for vector-valued functions we need the notion of a convex. However, the application of the mean-value theorem thus formulated is often inconvenient. In this paper I give a method which makes it sometimes possible to avoid applying the mean-value theorem. This may be obtained by introducing the uniform convergence with respect to a fundamental set of linear functionals. I was led to the idea of this notion through the study of the proof given by Alexiewicz and Orlicz in [3]. I wrote this paper under the direction of Professor W. Orlicz, whom I wish to thank for his help and valuable remarks.

Let X be a linear, normed and complete Banach space. Further let $\|x\|$ be the norm of the element $x \epsilon X$, Ξ the space of linear functionals over X and ξ an element of Ξ. $x(t)$, $x(s, t)$ indicate here vector-valued functions defined in the intervals $\langle a, b\rangle$ and $\langle a, b; c, d\rangle$ respectively, with values in the Banach space X.

We call the set Γ of linear functionals a *fundamental set* if there exist positive constants $a > 0, k > 0$ such that for every $\xi \epsilon \Gamma$ and $x \epsilon X$ the inequality

$$\sup |\xi x| \geqslant a\|x\|, \qquad \xi \epsilon \Gamma, \qquad \|\xi\| \leqslant K$$

holds. The unit sphere in the space Ξ, $\{\xi: \|\xi\| \geqslant 1\}$, is an example of a fundamental set of linear functionals.

The set Γ_0 of functionals will be called *strongly fundamental* if the condition

$$\sup_n |\xi x_n| < \infty \quad \text{for every} \quad \xi \epsilon \Gamma_0$$

implies

$$\overline{\lim_n} \|x_n\| < \infty.$$

It is known that every closed fundamental set of functionals is strongly fundamental. We shall indicate by Γ a fundamental set of functionals and by Γ_0 a strongly fundamental one.

The function $x(t)$ is *strongly differentiable at the point t_0 to the element* $x'(t_0)$ if

$$\left\| \frac{x(t_0+\Delta t)-x(t_0)}{\Delta t} - x'(t_0) \right\| \to 0 \quad \text{as} \quad \Delta t \to 0.$$

The function $x(t)$ is *weakly differentiable at the point t_0 to the element* $x'(t_0)$ if, for every $\xi \epsilon \Xi$,

$$\xi\left(\frac{x(t_0+\Delta t)-x(t_0)}{\Delta t} - x'(t_0)\right) \to 0 \quad \text{as} \quad \Delta t \to 0.$$

The sequence $\{x_n\}$ is *uniformly convergent with respect to the fundamental set Γ of linear functionals to the element x_0* if for every $\varepsilon > 0$ there exists a number N such that for an arbitrary integer $n > N$ and for each $\xi \epsilon \Gamma$ we have $|\xi x_n - \xi x_0| < \varepsilon$.

We say that the function $x(t)$ is *uniformly continuous with respect to the set Γ at the point t_0* if for arbitrary $\varepsilon > 0$ there exists a number $\delta > 0$ such that for every $\xi \epsilon \Xi$ and $|\Delta t| < \delta$

$$|\xi x(t_0+\Delta t) - \xi x(t_0)| < \varepsilon.$$

We call the function $x(t)$ *uniformly differentiable with respect to the set Γ at the point t_0 to the element $x'(t_0)$* if for arbitrary $\varepsilon > 0$ there exists a number $\delta > 0$ such that for every $|\Delta t| < \delta$ and $\xi \epsilon \Gamma$

$$\left| \xi \frac{x(t_0+\Delta t)-x(t_0)}{\Delta t} - \xi x'(t_0) \right| < \varepsilon.$$

Let $\Xi_0 \subset \Xi$. The function $x(t)$ is *Ξ_0-quasidifferentiable at the point t_0* if for every $\xi \epsilon \Xi_0$ there exists a derivative of the real-valued function $\xi x(t)$, $d\xi x(t)/dt$, at the point t_0.

LEMMA 1. *The sequence $\{x_n\}$ converges strongly to the element x_0 if and only if it is uniformly convergent to the element x_0 with respect to Γ.*

The proof of this lemma follows immediately from the definition of the fundamental set.

LEMMA 2. *Let us assume that for every $t \epsilon (a, b)$ there exists a Γ-quasiderivative $d\xi x(t)/dt$ and that for arbitrary $\varepsilon > 0$ there exists such a number $\delta > 0$, that for every $\xi \epsilon \Gamma$ and $|\Delta t| < \delta$ we have*

$$\left| \left(\frac{d\xi x(t)}{dt}\right)_{t+\Delta t} - \left(\frac{d\xi x(t)}{dt}\right)_{t} \right| < \varepsilon.$$

Then there exists in (a, b) a strong derivative $x'(t)$.

Proof. Let $t_0 \epsilon (a, b)$. Then the assumption of lemma 2 implies

$$\left| \xi \frac{x(t_0+h)-x(t_0)}{h} - \xi \frac{x(t_0+k)-x(t_0)}{k} \right| = \left| \left(\frac{d\xi x(t)}{dt}\right)_{t_0+v'h} - \left(\frac{d\xi x(t)}{dt}\right)_{t_0+v''k} \right| < \varepsilon$$

for every $\xi \epsilon \Gamma$. Hence, the completeness of the space X and lemma 1 imply the existence of an element $x'(t_0) \epsilon X$ such that there exists a strong limit

$$\lim_{\Delta t \to 0} \frac{x(t_0+\Delta t)-x(t_0)}{\Delta t} = x'(t_0).$$

COROLLARIES. 1. *A function is strongly continuous at the point t_0 if and only if it is uniformly continuous at the point t_0 with respect to Γ.*

2. *A function is strongly differentiable at the point t_0 if and only if it is uniformly differentiable at the point t_0 with respect to Γ.*

3. *If a function has in an enclosure of the point t_0 a weak derivative strongly continuous at the point t_0, then this function has a strong derivative at the point t_0.*

4. *If the function $x(t)$ has at the point t_0 a Γ_0-quasiderivative, then $x(t)$ is strongly continuous at the point t_0.*

Corollary 3 is a generalization of theorem 1 in [3] by Alexiewicz and Orlicz on the existence of the strong derivative.

APPLICATIONS

LEMMA 3. *Let us assume that the following conditions are satisfied:*

(a) *in an enclosure of (s_0, t_0) there exist weak derivatives x'_s, x'_t;*

(b) *there exists in an enclosure of (s_0, t_0) a Γ-quasiderivative with regard to the variable s, $\partial^2 \xi x/\partial t \partial s$, and given any $\varepsilon > 0$, there exists a number $\delta > 0$ such that for arbitrary $\xi \epsilon \Gamma$ and for every $|\Delta t| < \delta$, $|\Delta s| < \delta$, the inequality*

$$\left| \left(\frac{\partial^2 \xi x(s,t)}{\partial t \partial s}\right)_{t_0+\Delta t, s_0+\Delta s} - \left(\frac{\partial^2 \xi x(s,t)}{\partial t \partial s}\right)_{t_0, s_0} \right| < \varepsilon$$

holds.

Then there exist at the point (s_0, t_0) strong derivatives x''_{ts} and x''_{st} and $x''_{ts}(s_0, t_0) = x_{st}(s_0, t_0)$.

Proof. We have

$$\xi \frac{x(s_0+\Delta s, t_0+\Delta t) - x(s_0, t_0+\Delta t) + x(s_0+\Delta s, t_0) - x(s_0, t_0)}{\Delta s \Delta t} = \left(\frac{\partial^2(\xi x(s,t))}{\partial t \partial s}\right)_{s_0+\theta\Delta s_0, +\vartheta'\Delta t}.$$

The assumption implies

$$\left|\xi\frac{x(s_0+\Delta s_0,\,t_0+\Delta t_1)-x(s_0,\,t_0+\Delta t_1)+x(s_0+\Delta s_1,\,t_0)-x(s_0,\,t_0)}{\Delta s_1\Delta t_1}-\right.$$

$$\left.-\xi\frac{x(s_0+\Delta s_2,\,t_0+\Delta t_2)-(xs_0,\,t_0+\Delta t_2)+x(s_0+\Delta s_2,\,t_0)-x(s_0,\,t_0)}{\Delta s_0\,\Delta t_2}\right|\leqslant 2\varepsilon$$

for every $\xi\in\Gamma$, $|\Delta s_i|<\delta_i$, $|\Delta t_i|<\delta$, $i=1,2$.

Taking $\Delta s_i\to 0$, we have

$$\left|\frac{\xi x_s'(s_0,\,t_0+\Delta t_1)-\xi x_s'(s_0,\,t_0)}{\Delta t_1}-\frac{\xi x_s'(s_0,\,t_0+\Delta t_0)-\xi x_s'(s_0,\,t_0)}{\Delta t_2}\right|<2\varepsilon.$$

It follows that the strong limit

$$\lim_{\Delta t\to 0}\frac{x_s'(s_0,\,t_0+\Delta t)-x_s'(s_0,\,t_0)}{\Delta t}=x_{st}''(s_0,\,t_0)$$

exists.

In the same way it is possible to prove the existence of $x_{st}''(s_0,\,t_0)$. Evidently $x_{ts}''=x_{tx}''$.

SCHWARZ'S THEOREM. *If in an enclosure of $(s_0,\,t_0)$ the weak derivatives x_s', x_t', x_{ts}'' exist and the derivative x_{ts}'' is strongly continuous in the point $(s_0,\,t_0)$, then there exists a weak-strong derivative x_{st}'' and $x_{st}''(s_0,\,t_0)=$ $=x_{ts}''(s_0,\,t_0)$.*

Proof. The assumption of the existence of the strongly continuous weak derivative implies condition 2 of lemma 3.

L'HOPITAL'S RULE. *Let us assume that the vector-valued function $x(t)$ and the real-valued function $\gamma(t)$ are continuous in $\langle a,\,b)$, $\gamma'(t)$ exists in $(a,\,b)$, $x(a)=\Theta$ and $\gamma(a)=0$. Further let us assume that for every ξ exists the derivative $d\xi x(t)/dt$ and the left-side limit of the function*

$$\left(\frac{d\xi x(t)}{dt}\right)\Big/\gamma'(t)$$

at the point a[1]) *uniformly with respect to Γ.*

Then there exists a strong limit

$$\lim_{t\to a}\frac{x(t)}{\gamma(t)},$$

[1]) *I. e.*, for $0<t-a<\delta$ and $\xi\in\Gamma$, $\left|\dfrac{d\xi x(t)/dt}{\gamma'(t)}-g\right|<\varepsilon$.

i. e., there exists $y\in X$ such that

$$\left\|y-\frac{x(t)}{\gamma(t)}\right\|\to 0\quad \textit{for}\quad a<t\to a.$$

Proof. For sufficiently small $h>0$ and $k>0$ and for $\xi\in\Gamma$ we have

$$\left|\frac{\xi(x(a+h)-x(a))}{\gamma(a+h)-\gamma(a)}-\frac{\xi(x(a+k)-x(a))}{\gamma(a+k-\gamma(a)}\right|$$

$$\leqslant\left|\frac{d\xi x(a+\vartheta h)/dt}{\gamma'(a+\vartheta h)}-\lim_{t\to a+0}\frac{d\xi x(t)/dt}{\gamma'(t)}\right|$$

$$+\left|\frac{d\xi x(a+\vartheta' k)/dt}{\gamma'(a+\vartheta' k)}-\lim_{t\to a+0}\frac{d\xi x(t)/dt}{\gamma'(t)}\right|<2\varepsilon.$$

Hence

$$\lim_{h\to 0}\frac{x(a+h)}{\gamma(a+h)}=y\quad\text{where}\quad y\in X.$$

This theorem is a generalization of a theorem by Albrycht [1]. Albrycht obtains this theorem using stronger assumptions and applying in the proof the notion of a convex.

COROLLARY. 5. *If a function has a weak derivative, weakly continuous in $(a,\,b)$, then it has a strong derivative at each point except a set of the first category and measure zero.*

Proof. Corollary 4 implies that the function is strongly continuous. Applying the theorem by Alexiewicz and Orlicz [4], and in view of the fact that the set of the values of the derivative is separable, we obtain strong continuity of the derivative in a residual set. According to a theorem by Alexiewicz [2], we conclude that the set of points at which the function is not strongly differentiable is of measure zero.

6. *If the weak derivative exists and the set of its values is compact, then the derivative exists except a set of measure zero.*

7. *It the weak derivative is weakly continuous and the set of its values is compact, then the derivative exists in a strong sense.*

Corollaries 6 i 7 follow immediately from corollary 3 and from theorems by Alexiewicz and Orlicz [4].

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SUR L'UNICITÉ DE LA MOYENNE DE DOSS DES VARIABLES ALÉATOIRES SITUÉES DANS QUELQUES ESPACES DE BANACH

PAR

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1. Introduction. Mourier [3] a donné une définition de la moyenne pour une variable aléatoire située dans un espace de Banach. Doss [1] en a donné une autre pour des variables aléatoires situées dans un espace métrique (D). Il a appelé l'élément *a moyenne de la variable aléatoire* x si $(a, \lambda) \leqslant E(x, \lambda)$ quel que soit l'élément fixe $\lambda \epsilon (D)$, où (x, y) désigne la distance entre deux éléments quelconques x et y de (D), et E la moyenne classique d'un nombre aléatoire.

On sait que dans un espace de Banach, dans lequel les sphéroïdes sont mesurables, une moyenne quelconque de Mourier est aussi une moyenne de Doss.

La moyenne de Mourier a été déterminée [4] pour quelques classes de variables aléatoires situées dans les espaces de Banach suivants: l'espace (C) des fonctions continues dans [0,1], l'espace (c) des suites convergentes, l'espace (l) des suites $\{h_i\}$ telles que $\sum |h_i| < \infty$, et l'espace (L) des fonctions sommables dans [0,1], où certaines conditions sont imposées.

La moyenne de Doss d'une variable aléatoire située dans n'importe lequel des espaces mentionnés ci-dessus est par conséquent déterminée. L'unicité d'une telle moyenne est démontrée dans les théorèmes 1, 2, 3 et 4.

2. Nous aurons à nous appuyer sur les lemmes suivants:

LEMME 1. *Si le nombre aléatoire* x *est tel que* $E|x| < +\infty$, $\varepsilon > 0$ *étant donné, il existe un entier positif* n, *tel que*

$$|E(x)-\lambda| \leqslant E|x-\lambda| < |E(x)-\lambda| + 2\left(1+\frac{|\lambda|}{n}\right)\varepsilon;$$

où λ *est un nombre réel quelconque tel que* $[|\lambda|] = n \geqslant n_1$ ($[\lambda]$ *étant la partie entière du nombre réel* λ).