ON A PROBLEM OF BANACH

BY

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Banach [1] has asked for a characterization of those metric spaces which can be mapped in a continuous biunique fashion onto some compact metric space, and in particular has inquired whether the Banach space (c_0) can be mapped in this way.

Relevant results were obtained by Sikorski [7] and Katetov [3], but the case of (c_0) was not settled.

We remark here that *for* (c_0) *the answer is affirmative*, reasoning as follows: by a recent result of Kadec [2], (c_0) is homeomorphic with (l^1); by Mazur's theorem [6], (l^1) is homeomorphic with (l^2); by a theorem of the author [4], (l^2) is homeomorphic with its unit cell $C_n = \{x: \|x\| \leqslant 1\}$ (in the norm topology). Now the natural map of C_n onto C_w, the unit cell in the weak topology, is continuous, and C_w is known [5] to be homeomorphic with the Hilbert parallelotope P. Thus (c_0) admits a continuous biunique map onto P, which is a compact metric space, and the proof is complete.

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ON THE NOTION OF UNIFORM CONVERGENCE WITH RESPECT TO A FUNDAMENTAL SET OF FUNCTIONALS, AND ITS APPLICATION

BY

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When formulating the mean-value theorem for vector-valued functions we need the notion of a convex. However, the application of the mean-value theorem thus formulated is often inconvenient. In this paper I give a method which makes it sometimes possible to avoid applying the mean-value theorem. This may be obtained by introducing the uniform convergence with respect to a fundamental set of linear functionals. I was led to the idea of this notion through the study of the proof given by Alexiewicz and Orlicz in [3]. I wrote this paper under the direction of Professor W. Orlicz, whom I wish to thank for his help and valuable remarks.

Let X be a linear, normed and complete Banach space. Further let $\|x\|$ be the norm of the element $x \in X$, Ξ the space of linear functionals over X and ξ an element of Ξ. $x(t)$, $x(s, t)$ indicate here vector-valued functions defined in the intervals $\langle a, b\rangle$ and $\langle a, b; c, d\rangle$ respectively, with values in the Banach space X.

We call the set Γ of linear functionals a *fundamental set* if there exist positive constants $a > 0, k > 0$ such that for every $\xi \in \Gamma$ and $x \in X$ the inequality

$$\sup |\xi x| \geqslant a\|x\|, \qquad \xi \in \Gamma, \qquad \|\xi\| \leqslant K$$

holds. The unit sphere in the space Ξ, $\{\xi: \|\xi\| \geqslant 1\}$, is an example of a fundamental set of linear functionals.

The set Γ_0 of functionals will be called *strongly fundamental* if the condition

$$\sup_n |\xi x_n| < \infty \quad \text{for every} \quad \xi \in \Gamma_0$$

implies

$$\overline{\lim_n} \|x_n\| < \infty.$$

It is known that every closed fundamental set of functionals is strongly fundamental. We shall indicate by Γ a fundamental set of functionals and by Γ_0 a strongly fundamental one.