P R O B L È M E S

K. BORSUK (VARSOVIE)

P 144. Formulé dans la communication de J. Perkal, *Sur les ensembles ε-convexes*.

Ce fascicule, p. 9.

J. ACZÉL (DEBRECEN)

P 145. Formulé dans la communication *О теории средних*.

Ce fascicule, p. 40.

K. BORSUK (WARSAW)

P 146. A space M with a metric ϱ is said to be *strongly convex* if for every two points $a,b \in M$ there exists exactly one point $c \in M$ such that $\varrho(a,c)=\varrho(b,c)=\varrho(a,b)/2$. Does there exist for every polyhedron P a metrisation by which every point of P has a strongly convex neighbourhood in P?

Warsaw, 5. XI. 1954

P 147. Does there exist for every two polyhedrons A and B a finite system $f_1, f_2, \ldots, f_n$ of continuous mappings of A into B such that, for every space $X \supset A$, if there exists a continuous extension of f_i over X with respect to B for $i=1,2,\ldots,n$, then there exists also a continuous extension over X with respect to B of every continuous mapping of A into B?

Warsaw, 5. XI. 1954

B. KNASTER (WROCŁAW)

P 148. Appelons une décomposition continue en continus du cube à n dimensions

$$(*) \qquad I^n = \sum_{0 \leqslant s \leqslant 1} A_s$$

orthogonale à une autre décomposition continue en continus de ce cube,

$$I^n = \sum_{0\leqslant t\leqslant 1} B_t,$$

lorsque la décomposition

$$B_t = \sum_{0\leqslant s\leqslant 1} A_s \cdot B_t$$

est, pour tout t, continue en continus. La relation est symétrique.

Existe-t-il, pour $n>1$, une décomposition orthogonale à toute décomposition $(*)$ donnée d'avance?

P 149. Peut-il y avoir, dans une famille de décompositions continues en continus de I^n, plus que n qui soient orthogonales deux à deux?

Nouveau Livre Ecossais, Probl. 244 et 245, 12. XI. 1953

W. WOLIBNER (WROCŁAW)

P 150. Titchmarsh's theorem on convolution is equivalent to the following one: if the functions $f(x)$ and $g(x)$ $(0\leqslant x\leqslant\lambda)$ are integrable (L) and satisfy the equation

$$\int_0^x f(y)g(x-y)\,dy = 0, \qquad 0\leqslant x\leqslant\lambda,$$

then $f(x)$ or $g(x)$ vanishes almost everywhere for $0\leqslant x\leqslant\lambda/2$ [1]).

If f and g belong to a certain compact class K of equicontinuous functions (*e. g.* bounded by a number c and fulfilling Hölder condition with a fixed exponent and coefficient), then the preceding theorem implies the existence of a non-decreasing and continuous function $\varphi(\delta)$, $\delta\geqslant 0$, with $\varphi(0)=0$ such that if

$$\Big|\int_0^x f(y)g(x-y)\,dy\Big| \leqslant \delta, \qquad 0\leqslant x\leqslant\lambda,$$

then

$$\min\Big[\sup_{0\leqslant x\leqslant\lambda/2}|f(x)|,\ \sup_{0\leqslant x\leqslant\lambda/2}|g(x)|\Big] \leqslant \varphi(\delta).$$

Find out $\varphi(\delta)$ effectively for a certain compact class K.

Wrocław, 10. VI. 1954

[1]) J. G.-Mikusiński, *A new proof of Titchmarsh's theorem on convolutions*, Studia Mathematica 13 (1953), p. 56-58.

P 151. Let us consider the class of analytic univalent (*schlicht*) functions, having for $|z|>1$ the expansion of the form

$$f(z) = z + \sum_{n=1}^{\infty} \frac{b_n}{z^n}.$$

From Dieudonné's proof of Bieberbach conjecture for functions with real coefficients [2]) it follows directly, that if b_n are real, then

$$|b_n| \leqslant n(1-b_1), \qquad n=2,3,\ldots$$

(This estimation is better than

$$|b_n| \leqslant \sqrt{\frac{1-b_1^2}{n}},$$

which results from Bieberbach „Flächensatz" [3]), only if $1-b_1<2/(n^3+1)$).

The above estimation for $|b_n|/(1-b_1)$ cannot be strengthened, as it is seen from the following example:

$$f(z) = z + \frac{b_1}{z} - (1-b_1)\left[\frac{1/rz}{(1-1/rz)^2} - \frac{1}{rz}\right]$$

$$= z + \frac{b_1}{z} - \frac{2(1-b_1)}{r^2z^2} - \ldots - \frac{n(1-b_1)}{r^n z^n} - \ldots, \qquad 0<b_1<1,\ r>1$$

(this is a modified Koebe function). In fact, taking r sufficiently near to 1, we obtain $|b_n|/(1-b_1)$ arbitrarily little differing from n.

On the other hand, if, for a given r, b_1 is sufficiently near to 1, then this function is schlicht for $|z|>1$.

What is the estimation for $|b_n/(1-b_1)|$ when the coefficients of f are complex? This question seems to be interesting even for $n=2$.

Wrocław, 10. VI. 1954

P 152. Does there exist an analytical single-valued function fulfiling the following conditions:

1) it is a function of Pompeiu-Urysohn [4]) type, *i. e.* the set S of its essential singular points is non void, zero-dimensional and perfect,

[2]) J. Dieudonné, *Sur les fonctions univalentes*, Comptes Rendus (Paris), 192 (1931), p. 1148-1150.

[3]) L. Bieberbach, *Lehrbuch der Funktionentheorie II*, Leipzig-Berlin 1931, p. 72.

[4]) G. Pompeiu, *Sur la continuité des fonctions de variables complexes*, Annales de Toulouse (2), 7 (1905), p. 314, et P. Urysohn, *Sur une fonction partout continue*, Fundamenta Mathematicae 4 (1923), p. 144-150.

and at every point of S the function is defined and finite (hence it is a continuous function on the whole plane);

2) it possesses a simple pole in infinity;

3) it is univalent (schlicht) on the whole plane (hence it establishes an autohomeomorphism of the whole complex plane).

Wrocław, 9. XI. 1954

P 153. In the Analysis one often considers some sequences $\{P_n(x)\}$ of polynomials of increasing degree, with integral coefficients, *e. g.* 1° $P_n(x)=x^n-1$, 2° the sequence of Tchebychev polynomials, 3° the sequence of Legendre polynomials. The problem arises whether two polynomials of the same sequence P_n have a common root.

In 1° and 2° the problem is trivial, whereas in 3° for non-vanishing roots it is open and seems to be difficult.

For every sequence P_n the problem reduces to the question if for two polynomials of this sequence their discriminant vanishes. Hence it is a problem of number theory. From other difficult problems of number theory it distinguishes itself by its connexion with Analysis.

Wrocław, 20. XI. 1954

J. JAWOROWSKI (WARSAW)

P 154. Let L_k^n $(k<n)$ be the set of all k-dimensional hyperplanes of the projective n-dimensional space (the so-called *Grassmann manifold*). Is it possible to assign continuously to every $L\epsilon L_k^n$ a point belonging to L? For odd k – no, for even k and $n=k+1$ – yes. How is it in other cases (*e. g.* when $n=4$, $k=2$)?

New Scottish Book, Probl. 267, 1. IV. 1955

H. STEINHAUS (WROCŁAW)

P 155. The length of the circumference K is one, $\{P_n\}$ is a sequence of points on K. For every n the points $P_1, P_2, \ldots, P_n$ divide K into n arcs of lengths $l_k^{(n)}$ $(k=1,2,\ldots,n)$. The expression

$$W=\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n}(l_k^{(n)})^2$$

may be treated as a measure of the uniformity of the distribution of $\{P_n\}$ on K in the sense that, the less is W, the more uniformly distributed are the points P_n in K. The sequence $\{P_n\}$ is called *golden* if the length of the arc P_nP_{n+1} measured in a fixed direction is $(\sqrt{5}-1)/2$ for every n. A conjecture: The golden sequence gives the minimal value to W; the same value is given obviously by its symmetric image, and for every other sequence $\{P_n\}$ the number W is larger.

New Scottish Book, Probl. 268, 9. IV. 1955

JAN MYCIELSKI (WROCŁAW)

P 156. Let G be a free group. For any set $N\subset G$ we denote by $\{N\}$ the least subgroup of G containing the set N.

The condition

(*) φ^n non $\epsilon\{M-(\varphi)\}$ for every $\varphi\epsilon M$ and $n=1,2,3,\ldots$

is obviously necessary in order that M be a set of free generators for $\{M\}$.

Is (*) a sufficient condition also?

New Scottish Book, Probl. 260, 28. XII. 1954

A. RÉNYI (BUDAPEST)

P 157. Let α denote an irrational number. Let us consider the numbers $n\alpha-m$, where n and m run independently over all integers $(n,m=0,\pm1,\pm2,\ldots)$. Let us arrange the numbers $n\alpha-m$ in a sequence $x_k=n_k\alpha-m_k$, where (n_k,m_k) runs over all pairs of integers. For which arrangements is the sequence x_k equidistributed in the whole interval $(-\infty,+\infty)$? In other words, for which arrangements does the relation

$$\lim_{N\to\infty}\frac{\sum_{x_k\epsilon I_1, 1\leqslant k\leqslant N}1}{\sum_{x_k\epsilon I_2, 1\leqslant k\leqslant N}1}=\frac{|I_1I_2|}{|I_2|}$$

hold for any pair of intervals (I_1,I_2), where I_2 has a positive length (*i. e.* we suppose $|I_2|>0$, where $|I|$ denotes the length of the interval I)?

New Scottish Book, Probl. 262, 30. XII. 1954

P 158. Let $\Pi=(p_{jk})$ $(j,k=1,2,\ldots)$ denote a double stochastic infinite matrix, all elements of which are positive, *i. e.* we suppose $p_{jk}>0$ $(j,k=1,2,\ldots)$,

$$\sum_{k=1}^{\infty}p_{jk}=1 \quad (j=1,2,\ldots), \quad \text{and} \quad \sum_{j=1}^{\infty}p_{jk}=1 \quad (k=1,2,\ldots).$$

Let us put $\prod^n = (p_{jk}^{(n)})$ $(n=1,2,\ldots)$, where $\prod^n$ denotes the n-th power of the matrix $\prod$. Does the relation

$$\lim_{n\to\infty} \frac{p_{jk}^{(n)}}{p_{il}^{(n)}} = 1 \qquad (i,j,k,l=1,2,\ldots)$$

hold for all matrices of the considered type?

New Scottish Book, Probl. 263, 30. XII. 1954

P 159. Let $Q(n,k)$ denote the shortest sequence which can be formed of the numbers $1,2,\ldots,n$ and which has the property, that any two integers i and j $(1 \leqslant i < j \leqslant n)$ occur in the sequence in such a position, that they are separated by less then k elements of the sequence (*i. e.* they are neighbours, second neighbours,... k-th neighbours). Does the limit

$$\lim_{n\to\infty} \frac{Q(n,k)}{n^2}$$

exist for every $k=1,2,3,\ldots$?

Remark. I have proved the existence of this limit for $k=1$; for $k=2$ the existence of the limit has been proved by N. N. de Bruijn (Amsterdam). For $k=1$ or 2 the limit is $1/2k$. For arbitrary k I can prove only

$$\frac{1}{2k} \leqslant \varliminf_{n\to\infty} \frac{Q(n,k)}{n^2} \leqslant \varlimsup_{n\to\infty} \frac{Q(n,k)}{n^2} \leqslant \frac{1}{k+1}.$$

New Scottish Book, Probl. 264, 30. XII. 1954

P 159, R1. Since the problem has been proposed, we have proved with P. Erdös the existence of the limit in question for all values of k. The proof will be published in our joint paper *On some combinatorical problems* (in print in Publicationes Mathematicae). The value of the limit $\lim_{n\to\infty} Q(n,k)/n^2$ for $k>2$ is still unknown.

Added in proof