Similarly, when we cancel in the matrix $\bar{A}$ the column j_1 for which

$$\sum_i a_{ij_1} x_i^0 < M(\bar{A})$$ [2])

we get the matrix $\bar{A}'$ which satisfies the equation $M(\bar{A}') = M(\bar{A})$. In virtue of (10) and (6) we also have the inequality

(11) $$M(\bar{A}') = M(\bar{A}) \geqslant M(A) \geqslant m(A) = m(\bar{A}) \geqslant m(\bar{A}').$$

Hence

(11a) $$M(\bar{A}') \geqslant M(A) \geqslant m(A) \geqslant m(\bar{A}').$$

Repeating, if necessary, the above process of cancelation of rows and columns, we finally get the matrix $B = \{b_{ij}\}$, which satisfies the inequality

(11b) $$M(B) \geqslant M(A) \geqslant m(A) \geqslant m(B)$$

and the equations [3])

$$\sum_j a_{ij} y_j^0 = m(B), \qquad \sum_i a_{ij} x_i^0 = M(B)$$

$$(i = 1, 2, \ldots, p' \leqslant p;\ j = 1, 2, \ldots, q' \leqslant q).$$

From these equations we immediately find $m(B) = M(B)$, as the left sides of these equations are equal to

$$\sum_{ij} a_{ij} x_i^0 y_j^0,$$

and hence considering (11b), we get the theorem.

[2]) x^0 is an extremal point for the matrix $\bar{A}$.
[3]) y^0 is an extremal point for the matrix B.

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ON THE GAME OF BANACH AND MAZUR

BY

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In this note*) I am speaking about a game which H. Steinhaus calls a *game of Banach and Mazur*. This game is defined in the following way.

On an infinite half-line $0 \leqslant x \leqslant \infty$ a set Z is given. There are two players, A and B. Player A begins the play by choosing, in the first move, a positive number a_1. Subsequently in the second move, the player B chooses a positive number b_1 smaller than a_1. Then, in the third move, the player A chooses a positive number a_2 smaller than b_1. They do so by turns infinitely many times. When the play is finished, an infinite decreasing sequence

(1) $$a_1 > b_1 > a_2 > b_2 > \ldots$$

of positive numbers is obtained. In this sequence the numbers a_i are chosen by the player A and numbers b_i are chosen by the player B. If the number

$$g = \sum_{i=1}^{\infty} (a_i + b_i)$$

is in the set Z, the player A wins, if it is not in the set Z, the player B wins.

In other words, the player A chooses a function a which, for each n, given the numbers $a_1, b_1, \ldots, a_{n-1}, b_{n-1}$, prescribes the value of a_n. The player B chooses an analogous function b which, for each n, given the numbers $a_1, b_1, \ldots, b_{n-1}, a_n$, prescribes the value of b_n. Each choice is made in complete ignorance of the others. The functions a and b are called *strategies*. They determine the sequence (1) and therefore the winner.

In the theory of games, a game is called *closed* [1]) if for one of the players there exists a strategy which makes him win, no matter what strategy is used by his opponent.

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[1]) This definition was first given in [3]. In [1] the term "determined game" is used.

It is intuitively felt that for small sets Z the game of Banach and Mazur should be closed to the advantage of the player B. Now the question arises for what sets Z this game is really closed to the advantage of the player B. Turowicz [4] has shown that it is so if Z is the set of rational numbers. In this note I wish to generalize his result by proving the following [2])

THEOREM. *If Z is a denumerable set, then the game of Banach and Mazur is closed to the advantage of the player B.*

Proof. Let us arrange the elements of the set Z in a sequence $z_1, z_2, \ldots$. For each natural k let us denote by s_k the sum $a_1+b_1+a_2+b_2+\ldots+b_{k-1}+a_k$. We shall prove the theorem by defining a method of choosing the numbers b_i in order to have $g \neq z_i$ for each i. It is the following method:

The player B decides at the $2k$-th move of the play (that is at his k-th move) what inequalities must be fulfilled by the numbers $b_k, b_{k+1}, b_{k+2}, \ldots$ in order to have $g \neq z_k$. These inequalities, imposed upon the numbers $b_k, b_{k+1}, b_{k+2}, \ldots$ in the $2k$-th move, depend on the sum s_k and the number z_k. Namely, the sum s_k being given, the player B chooses the positive numbers $\beta_k^{(k)}, \beta_{k+1}^{(k)}, \ldots$ so that, if $s_k < z_k$, we have the inequality

$$\sum_{i=k}^{\infty} \beta_i^{(k)} < (z_k - s_k)/2; \tag{2}$$

if $z_k \leqslant s_k$, he puts, for instance, $1 = \beta_k^{(k)} = \beta_{k+1}^{(k)} = \ldots$ Then he chooses b_n, for $n = k, k+1, \ldots$ so that the inequality

$$b_n < \beta_n^{(k)} \tag{3}$$

holds. Thus, if $s_k < z_k$, then, in view of (1) and (2), we shall have

$$g = s_k + (b_k + a_{k+1}) + (b_{k+1} + a_{k+2}) + \ldots$$
$$< s_k + 2\sum_{i=k}^{\infty} \beta_i^{(k)} < s_k + 2\cdot \frac{z_k - s_k}{2} = z_k,$$

and, if $z_k \leqslant s_k$ we shall have $z_k < g$ in virtue of the positiveness of the numbers a_n and b_n, and thus in both cases we shall have $g \neq z_k$.

According to the method described, the player B has to choose his k-th number, b_k, so that the following $k+1$ inequalities be fulfilled:

$$b_k < a_k,\ b_k < \beta_k^{(1)}, \ldots, b_k < \beta_k^{(k)},$$

[2]) Recently M. Reichbach [2] has shown that there exists a perfect set Z of measure zero for which the game of Banach and Mazur is closed, but to the advantage of the player A.

the first of them following from the definition of the game and the other k being imposed by the player B himself in his first, second, ..., k-th move respectively.

Since the player B, according to the method described, ensures the inequality $g \neq z_k$ in his k-th move, we shall have $g \neq z_i$ for every i. The theorem is proved.

Note that in this proof we have indicated a whole class of winning strategies of the player B.

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