Similarly, when we cancel in the matrix $\tilde{A}$ the column $j_1$ for which
$$
\sum_j a_{i_j}a_{i_1}^j < M(\tilde{A})
$$
we get the matrix $\tilde{A}'$ which satisfies the equation $M(\tilde{A}') = M(\tilde{A})$. In virtue of (10) and (9) we also have the inequality
$$
M(\tilde{A}') = M(\tilde{A}) \geq m(A) \geq m(\tilde{A}) \geq m(\tilde{A}').
$$
Hence
$$
M(\tilde{A}') \geq M(\tilde{A}) \geq m(A) \geq m(\tilde{A}').
$$

Repeating, if necessary, the above process of cancelation of rows and columns, we finally get the matrix $B = [b_{i_j}]$, which satisfies the inequality
$$
M(B) \geq M(A) \geq m(A) \geq m(B)
$$
and the equations
$$
\sum_i a_{i_j}y_i = m(B), \quad \sum_j a_{i_j}a_{i_1}^j = M(B)
$$

(i = 1, 2, ..., $p'; j = 1, 2, ..., q'; l \leq q$).

From these equations we immediately find $m(B) = M(B)$, as the left sides of these equations are equal to
$$
\sum_j a_{i_j}a_{i_1}^j y_i,
$$
and hence considering (11b), we get the theorem.

---

1) $a^p$ is an extremal point for the matrix $A$.
2) $y^p$ is an extremal point for the matrix $B$.

---

ON THE GAME OF BANACH AND MAZUR

BY

S. ZUBRYCKI (WROCLAW)

In this note I am speaking about a game which H. Steinhaus calls a game of Banach and Mazur. This game is defined in the following way.

On an infinite half-line $0 \leq x \leq \infty$ a set $Z$ is given. There are two players, $A$ and $B$. Player $A$ begins the play by choosing, in the first move, a positive number $a_1$. Subsequently in the second move, the player $B$ chooses a positive number $b_1$ smaller than $a_1$. Then, in the third move, the player $A$ chooses a positive number $a_2$ smaller than $b_1$. They do so by turns infinitely many times. When the play is finished, an infinite decreasing sequence
$$
a_1 > b_1 > a_2 > b_2 > \ldots
$$
of positive numbers is obtained. In this sequence the numbers $a_i$ are chosen by the player $A$ and numbers $b_i$ are chosen by the player $B$. If the number
$$
y = \sum_{i=1}^{\infty} (a_i + b_i)
$$
is in the set $Z$, the player $A$ wins, if it is not in the set $Z$, the player $B$ wins.

In other words, the player $A$ chooses a function $a$ which, for each $n$, given the numbers $a_1, b_1, \ldots, a_{n-1}, b_{n-1}$, prescribes the value of $a_n$. The player $B$ chooses an analogous function $b$ which, for each $n$, given the numbers $a_1, b_1, \ldots, a_{n-1}, b_{n-1}$, prescribes the value of $b_n$. Each choice is made in complete ignorance of the others. The functions $a$ and $b$ are called strategies. They determine the sequence (1) and therefore the winner.

In the theory of games, a game is called closed if for one of the players there exists a strategy which makes him win, no matter what strategy is used by his opponent.

*1) Presented to the Polish Mathematical Society, Section of Wroclaw, the 15. X. 1964.
2) This definition was first given in [2]. In [1] the term "determined game" is used.
It is intuitively felt that for small sets Z the game of Banach and Mauro should be closed to the advantage of the player B. Now the question arises for what sets Z this game is really closed to the advantage of the player B. Turovic [4] has shown that it is so if Z is the set of rational numbers. In this note I wish to generalize his result by proving the following:

**Theorem.** If Z is a denumerable set, then the game of Banach and Mauro is closed to the advantage of the player B.

Proof. Let us arrange the elements of the set Z in a sequence \( z_1, z_2, \ldots \). For each natural \( k \) let us denote by \( b_k \) the sum \( a_1 + a_2 + \ldots + a_{k-1} + a_k \). We shall prove the theorem by defining a method of choosing the numbers \( b_k \) in order to have \( g \neq a_k \) for each \( k \). It is the following method:

The player B decides at the 2k-th move of the play (that is at his \( k \)-th move) what inequalities must be fulfilled by the numbers \( b_{2k}, b_{2k+1} \), \( b_{2k+2}, \ldots \) in order to have \( g \neq a_k \). These inequalities, imposed upon the numbers \( b_{2k}, b_{2k+1}, b_{2k+2}, \ldots \) in the 2k-th move, depend on the sum \( s_k \) and the number \( a_k \). Namely, the sum \( s_k \) being given, the player B chooses the positive numbers \( \beta_k, \beta_{k+1}, \ldots \) so that, if \( s_k < a_k \), we have the inequality

\[
\sum_{k=1}^{\infty} \beta_k < \frac{(s_k - a_k)}{2};
\]

if \( s_k \geq a_k \), he puts, for instance, \( 1 = \beta_k = \beta_{k+1} = \ldots \). Then he chooses \( b_k \) for \( n = k, k+1, \ldots \) so that the inequality

\[
b_k < \beta_k
\]

holds. Thus, if \( s_k < a_k \), then, in view of (1) and (2), we shall have

\[
g = s_k + (a_k + a_{k+1} + \ldots + (b_{2k+1} + a_{2k+2} + \ldots
\]

\[
< s_k + \sum_{k=1}^{\infty} \beta_k < a_k + 2 \cdot \frac{s_k - a_k}{2} = z_k,
\]

and, if \( s_k \geq a_k \) we shall have \( s_k < g \) in virtue of the positiveness of the numbers \( a_k \) and \( b_k \), and thus in both cases we shall have \( g \neq a_k \).

According to the method described, the player B has to choose his \( k \)-th number, \( b_k \), so that the following \( k+1 \) inequalities be fulfilled:

\[
b_k < a_k, b_k < \beta_k, \ldots, b_k < \beta_{k+1};
\]

the first of them following from the definition of the game and the other \( k \) being imposed by the player B himself in his first, second, \ldots, \( k \)-th move respectively.

Since the player B, according to the method described, ensures the inequality \( g \neq a_k \) in his \( k \)-th move, we shall have \( g \neq a_i \) for every \( i \). The theorem is proved.

Note that in this proof we have indicated a whole class of winning strategies of the player B.

**References**


**Mathematical Institute of the Polish Academy of Sciences**