AN ELEMENTARY PROOF OF VON NEUMANN'S MINIMAX THEOREM

BY

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Let $A = a_{ij}$ $(i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, q)$ be a matrix of real numbers. Let us put

$$E(x, y) = \sum_{i,j} a_{ij} x_i y_j$$

where x, y stands for systems of real variables $(x_1, x_2, \ldots, x_p)$, $(y_1, y_2, \ldots, y_q)$. Further let X_p and Y_q be sets of x's and y's which satisfy the following relations:

$$\sum_i x_i = 1, \quad \sum_j y_j = 1 \quad (x_1 \geqslant 0, \ldots, x_p \geqslant 0;\ y_1 \geqslant 0, \ldots, y_q \geqslant 0).$$

Our aim is to give an elementary proof of the following equation (so-called *von Neumann's minimax theorem* of the theory of games)[1]):

$$(1) \qquad \min_{x\epsilon X_p} \max_{y\epsilon Y_q} E(x, y) = \max_{y\epsilon Y_q} \min_{x\epsilon X_p} E(x, y).$$

Let us denote the left side of the equation (1) by $M(A)$ and the right side by $m(A)$. It is clear that

$$(2) \qquad M(A) \geqslant m(A).$$

If $\bar{A}$ is a matrix obtained from a given matrix A by cancelling a row, and A' a matrix obtained from A by cancelling a column, then we have the following obvious inequalities:

$$(3) \quad M(\bar{A}) \geqslant M(A), \qquad (4) \quad m(\bar{A}) \geqslant m(A),$$

$$(5) \quad M(A') \leqslant M(A), \qquad (6) \quad m(A') \leqslant m(A)$$

(after the cancelation of a row or a column the ranges of variability of the indices i, j are changed).

[1]) See *e. g.* J. C. C. McKinsey, *Introduction to the theory of games*, New York 1952, Theorem 2.6, p. 34.

Let $x^0 = (x_1^0, x_2^0, \ldots, x_p^0)$ and $y^0 = (y_1^0, y_2^0, \ldots, y_q^0)$ denote any two extremal points i.e. points which satisfy the equations

$$\max_{y\epsilon Y_q} E(x^0, y) = M(A); \qquad \min_{x\epsilon X_p} E(x, y^0) = m(A).$$

Such points exist because the form $E(x, y)$ is continuous and the spaces X_p, Y_q are closed.

Let us cancel in the matrix A one row i_1 from rows (if such rows exist) which fulfil the inequality

$$(7) \qquad \sum_j a_{i_1 j} y_j^0 > m(A).$$

We show, that the matrix $\bar{A}$ thus obtained satisfies the equation

$$(8) \qquad m(A) = m(\bar{A}).$$

If (8) did not hold, then considering (4) we should have a system $y^0 + \Delta y = (y_1^0 + \Delta y_1, \ldots, y_q^0 + \Delta y_q)$ such that

$$(9) \qquad \min_{x\epsilon X_{p-1}} E(x, y^0 + \Delta y) > \min_{x\epsilon Xp} E(x, y^0) = m(A).$$

(The variable x on the left side of this formula runs through a subspace X_{p-1} of the space X_p which appears on the right side, namely systems of the type $(X_1, \ldots, X_{i_1-1}, 0, X_{i_1+1}, \ldots, X_p)$ belong to X_{p-1}).

Considering that the form $E(x, y)$ is bilinear, the inequality

$$(9a) \qquad \min_{x\epsilon X_{p-1}} E(x, y^0 + \varepsilon \Delta y) > \min_{x\epsilon X_p} E(x, y^0)$$

holds for every value of ε, $0 < \varepsilon \leqslant 1$.

On the other hand in view of (7) for sufficiently small ε, we have the inequality

$$(7a) \qquad \sum_j a_{i_1 j}(y_j^0 + \varepsilon \Delta y_j) > m(A);$$

but, considering (9a) and (7a), we get

$$\min\big[\min_{x\epsilon X_{p-1}} E(x, y^0 + \epsilon \Delta y),\ \sum_j a_{i_1 j}(y_j^0 + \varepsilon \Delta y_j)\big]$$
$$= \min_{x\epsilon X_p} E(x, y^0 + \epsilon \Delta y) > \min_{x\epsilon X_p} E(x, y^0),$$

which is contrary to the definition of y^0.

Now, considering (8), (3) and (2), we have

$$(10) \qquad M(\bar{A}) \geqslant M(A) \geqslant m(A) = m(\bar{A}).$$

Similarly, when we cancel in the matrix $\bar{A}$ the column j_1 for which

$$\sum_i a_{ij_1} x_i^0 < M(\bar{A})$$ [2])

we get the matrix $\bar{A}'$ which satisfies the equation $M(\bar{A}') = M(\bar{A})$. In virtue of (10) and (6) we also have the inequality

(11) $$M(\bar{A}') = M(\bar{A}) \geqslant M(A) \geqslant m(A) = m(\bar{A}) \geqslant m(\bar{A}').$$

Hence

(11a) $$M(\bar{A}') \geqslant M(A) \geqslant m(A) \geqslant m(\bar{A}').$$

Repeating, if necessary, the above process of cancelation of rows and columns, we finally get the matrix $B = \{b_{ij}\}$, which satisfies the inequality

(11b) $$M(B) \geqslant M(A) \geqslant m(A) \geqslant m(B)$$

and the equations[3])

$$\sum_j a_{ij} y_j^0 = m(B), \qquad \sum_i a_{ij} x_i^0 = M(B)$$

$$(i = 1, 2, \ldots, p' \leqslant p;\ j = 1, 2, \ldots, q' \leqslant q).$$

From these equations we immediately find $m(B) = M(B)$, as the left sides of these equations are equal to

$$\sum_{ij} a_{ij} x_i^0 y_j^0,$$

and hence considering (11b), we get the theorem.

[2]) x^0 is an extremal point for the matrix $\bar{A}$.
[3]) y^0 is an extremal point for the matrix B.

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ON THE GAME OF BANACH AND MAZUR

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In this note*) I am speaking about a game which H. Steinhaus calls a *game of Banach and Mazur*. This game is defined in the following way.

On an infinite half-line $0 \leqslant x \leqslant \infty$ a set Z is given. There are two players, A and B. Player A begins the play by choosing, in the first move, a positive number a_1. Subsequently in the second move, the player B chooses a positive number b_1 smaller than a_1. Then, in the third move, the player A chooses a positive number a_2 smaller than b_1. They do so by turns infinitely many times. When the play is finished, an infinite decreasing sequence

(1) $$a_1 > b_1 > a_2 > b_2 > \ldots$$

of positive numbers is obtained. In this sequence the numbers a_i are chosen by the player A and numbers b_i are chosen by the player B. If the number

$$g = \sum_{i=1}^{\infty} (a_i + b_i)$$

is in the set Z, the player A wins, if it is not in the set Z, the player B wins.

In other words, the player A chooses a function a which, for each n, given the numbers $a_1, b_1, \ldots, a_{n-1}, b_{n-1}$, prescribes the value of a_n. The player B chooses an analogous function b which, for each n, given the numbers $a_1, b_1, \ldots, b_{n-1}, a_n$, prescribes the value of b_n. Each choice is made in complete ignorance of the others. The functions a and b are called *strategies*. They determine the sequence (1) and therefore the winner.

In the theory of games, a game is called *closed*[1]) if for one of the players there exists a strategy which makes him win, no matter what strategy is used by his opponent.

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[1]) This definition was first given in [3]. In [1] the term "determined game" is used.