Theorem 1 leads to the following problems, which are open also in the algebraic case (8):

P 163. Does there exist

$$\lim_{n\to\infty}\frac{\varepsilon_n(\xi;T)}{\varepsilon_n(\xi)}\ ?$$

(We know only that $1\leqslant \varepsilon_n(\xi;T)/\varepsilon_n(\xi)\leqslant s<+\infty$ for any $n\geqslant m$.)

P 164. Is

$$\lim_{n\to\infty}\frac{\varepsilon_n(\xi;T)}{\varepsilon_n(\xi)}=1\ ?$$

If the answer to both problems is positive, then

$$\lim_{n\to\infty}\frac{\varepsilon_n(\xi;T)}{\varepsilon_n(\xi)}=1,$$

which would very essentially strengthen theorem 1.

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ON THE NUMBER OF AFFINICALLY DIFFERENT SETS

BY

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The problem solved in this paper*) concerns the constructive theory of functions. In the theory of uniform approximation the notion of the *polynomial of the best approximation* of a continuous function $x(t)$ in the interval $\langle a,b\rangle$ plays an important role. Such a polynomial of degree n is defined in [1] as a polynomial realising

$$\min_{w\in W_n}\max_{a\leqslant t\leqslant b}|x(t)-w(t)|,$$

where W_n is the class of all polynomials of a degree not greater than n. Utilizing some results of de la Vallée Poussin [2], Rémès has given in [3] a recurrently defined sequence of polynomials, convergent to the polynomial of the best approximation. De la Vallée Poussin has remarked that the polynomial $v(t)$ of degree n which gives the best approximation of a function $x(t)$ on the system of $n+2$ points $t_0, t_1, \ldots, t_{n+1}$ $(t_0<t_1<\ldots<t_{n+1})$ satisfies the system of $n+2$ equations

$$v(t_i)+(-1)^i\varepsilon = x(t_i)\qquad (i=0,1,\ldots,n+1)$$

(ε is here the $(n+2)$-th unknown). Thus, if $v(t)=a_0+a_1t+\ldots+a_nt^n$, then

$$a_0+a_1t_0+\ldots+a_nt_0^n+\varepsilon = x(t_0),$$

$$a_0+a_1t_1+\ldots+a_nt_1^n-\varepsilon = x(t_1),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_0+a_1t_{n+1}+\ldots+a_nt_{n+1}^n+(-1)^{n+1}\varepsilon = x(t_{n+1}).$$

Thus any coefficient of the polynomial $v(t)$ is a linear combination of the value of the function $x(t)$ with the coefficients depending only on the points $t_0, t_1, \ldots, t_{n+1}$:

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$$a_k = a_{k,0}(t_0, t_1, \ldots, t_{n+1})x(t_0) + a_{k,1}(t_0, t_1, \ldots, t_{n+1})x(t_1) + \ldots$$
$$+ a_{k,n+1}(t_0, t_1, \ldots, t_{n+1})x(t_{n+1}) \qquad (k = 0, 1, \ldots, n).$$

If the points $t_0, t_1, \ldots, t_{n+1}$ belong to a fix finite set, *e. g.* — as in the sequel — to the set of non-negative integers $0, 1, \ldots, s$, then for a simplified calculation of the polynomial $v(t)$ (and — by the result of Rémès — for a calculation of the polynomial of the best approximation) we can tabulate the coefficients

(1) $$a_{0,0}(t_0, t_1, \ldots, t_{n+1}), \ldots, a_{n,n+1}(t_0, t_1, \ldots, t_{n+1})$$

for different systems $\{t_0, t_1, \ldots, t_{n+1}\}$. But if the system $\{t_0^*, t_1^*, \ldots, t_{n+1}^*\}$ can be transformed affinically into the system $\{t_0, t_1, \ldots, t_{n+1}\}$, $t_i = at_i^* + \beta$ $(i = 0, 1, \ldots, n+1)$, then the polynomials $v^*(t)$ and $v(t)$ corresponding to these systems are combined by the analogical formula,

$$v(at+\beta) = v^*(t).$$

Hence, it is not necessary to tabulate the function (1) for all subsets (consisting of $n+2$ points) of the set $\{0, 1, \ldots, s\}$, but only for a system of subsets which cannot be affinically transformed into one another. In order to obtain the volume of such tables of the function (1) we must therefore solve the following problem:

On a straight line there is given a set A_s consisting of points with coordinates $0, 1, \ldots, s$. $\mathcal{A}_{s,r}$ denotes the family of all subsets of A_s consisting of $r+1$ points $(r \geqslant 1)$. $\mathcal{A}_{s,r}$ is divided into classes: two sets belong to the same class if and only if one of them can be affinically transformed into the other. The formula by which we may calculate the number $a_{s,r}$ of those classes is to be found.

Two sets are said to be *affinically different* if one of them cannot be affinically transformed into the other.

The solution will be divided into several parts.

1. By $a_{k,r}$ we denote the number of subsets of the set $A_k = \{0, 1, \ldots, k\}$ which consist of $r+1$ points and have the diameter k.

$$a_{k,r} = \binom{k-1}{r-1},$$

because the first and the last point of any of those subsets are the same for all of them, namely 0 and k.

2. Now we shall calculate the number denoted by $b_{k,r}$ of subsets $\{a_0, a_1, \ldots, a_r\}$ of the set A_k which consist of $r+1$ points, have the diameter k, and are symmetrical with respect to the point $k/2$, *i. e.* with respect to the centre of the interval $(0, k)$.

2.1. k and r are odd: $k = 2\varkappa-1$ and $r = 2\varrho-1$ (here and in the sequel $\varkappa$ and ϱ are natural numbers). Hence $a_0 = 0$, the elements with indices $1, 2, \ldots, \varrho-1$ belonging to this subset belong to the set $\{1, 2, \ldots, \varkappa-1\}$, and the other elements are obtained by the reflexion in the point $k/2$. Thus

$$b_{k,r} = \binom{\varkappa-1}{\varrho-1}.$$

2.2. k is odd, r is even. In this case there are no symmetrical subsets:

$$b_{k,r} = 0.$$

2.3. k is even, r is odd, *i. e.* $k = 2\varkappa$ and $r = 2\varrho-1$. As in the first case

$$b_{k,r} = \binom{\varkappa-1}{\varrho-1},$$

because in the set $\{0, 1, \ldots, k\}$ there exists an element $\varkappa$ in the middle which can belong to none of the symmetrical subsets with r odd and because the elements of the subset with indices $1, 2, \ldots, \varrho-1$ now belong to the set $\{1, 2, \ldots, \varkappa-1\}$.

2.4. k and r are odd, *i. e.* $k = 2\varkappa$ and $r = 2\varrho$. In this case the middle element of the subset with index ϱ is the number $\varkappa$. The elements with indices $1, 2, \ldots, \varrho-1$ now belong to the set $\{1, 2, \ldots, \varkappa-1\}$:

$$b_{k,r} = \binom{\varkappa-1}{\varrho-1}.$$

Taking into account the values of $\varkappa$ and ϱ, we obtain

$$b_{k,r} = \chi(k,r)\binom{-[-k/2]-1}{-[-r/2]-1} = \chi(k,r)\binom{-[1-k/2]}{-[1-r/2]},$$

where

$$\chi(k,r) = \begin{cases} 0 & \text{if } k \text{ is odd and } r \text{ even,} \\ 1 & \text{in other cases,} \end{cases}$$

and $[x]$ denotes the integral part of x.

3. By 1 and 2 we find that the number of subsets of the set A_k, with $r+1$ points, of diameter k, which are affinically different, is

$$c_{k,r} = \frac{a_{k,r}-b_{k,r}}{2} + b_{k,r} = \frac{a_{k,r}+b_{k,r}}{2}.$$

4. From the subsets consisting of $r+1$ points, of diameter k, of the set A_k, *i. e.* subsets of the form $\{0 = a_0, a_1, \ldots, a_r = k\}$, we exclude those which can be obtained from the subsets of a smaller diameter by similarity (dilatation). Such a dilatation extending to covering the whole set A_k (*i. e.* the first element 0 and the last k) is possible only if k is a composite number. Let k be such a number, $d|k$, and $d>1$. The set $\{0, d, 2d, \ldots, (k/d)d = k\}$ can be obtained by a d-fold dilatation of the set $E = \{0, 1, 2, \ldots, k/d\}$. The number of subsets of E which consist of $r+1$ points, are affinically different and have diameter k/d, is (according to 3)

$$c_{k/d,r} = \frac{a_{k/d,r}+b_{k/d,r}}{2}.$$

Evidently we must have here $k/d \geqslant r$, *i. e.* $d \leqslant k/r$.

We use now the Möbius function $\mu(n)$, which is defined by the formulae

$$\mu(1) = 1, \qquad \mu(n) = \begin{cases} 0 & \text{if} \quad p^2|n, \\ (-1)^j & \text{if} \quad n = p_1p_2\ldots p_j. \end{cases}$$

($p, p_1, \ldots, p_j$ denote prime numbers, $p_1, \ldots, p_j$ are all different from one another). This function satisfies for $m>1$ the equation

$$\sum_{d|m} \mu(d) = 0,$$

which may be written in the form

$$1 = -\sum_{1<d|m} \mu(d). \tag{2}$$

Let $d_{k,r}$ be the number of affinically different subsets of diameter k of the set A_k which can be obtained by a dilatation of the subsets of a smaller diameter. The subsets of diameter k are denoted by $B_1, B_2, \ldots, B_{d_{k,r}}$. δ_i is the greatest divisor of the number k such that the set B_i can be obtained by a δ_i-fold dilatation of a subset of a smaller diameter than k. We write $d_{k,r}$ equalities following from (2):

$$1 = -\sum_{1<d|\delta_i} \mu(d) \qquad (i = 1, 2, \ldots, d_{k,r}). \tag{3}$$

Adding them we get

$$d_{k,r} = -\sum_{d|k, 1<d\leqslant k/r} \mu(d)c_{k/d,r},$$

because $\mu(d)$ occurs in the formulae (3) as many times as there are sets $B_1, B_2, \ldots, B_{d_{k,r}}$ obtainable by a d-fold dilatation from the sets of diameter k/d.

5. If from all the affinically different sets of diameter k we exclude those which after a contraction can have a smaller diameter, and if there numbers are added for all k, we obtain the formula

$$a_{s,r} = \sum_{k=r}^{s} (c_{k,r} - d_{k,r}),$$

representing the number of classes into which the family $\mathcal{A}_{s,r}$ has been divided.

On transforming the last formula, we obtain

$$a_{s,r} = \sum_{k=r}^{s}\Big(c_{k,r} + \sum_{d|k,1<d\leqslant k/r} \mu(d)c_{k/d,r}\Big) = \sum_{k=r}^{s}\sum_{d|k,d\leqslant k/r} \mu(d)c_{k/d,r}$$

$$= \frac{1}{2}\sum_{k=r}^{s}\sum_{d|k,d\leqslant k/r} \mu(d)(a_{k/d,r}+b_{k/d,r}),$$

and finally

$$a_{s,r} = \frac{1}{2}\sum_{k=r}^{s}\sum_{d|k,d\leqslant k/r} \mu(d)\left\{\binom{k/d-1}{r-1} + \chi(k/d, r)\binom{-[1-k/2d]}{-[1-r/2]}\right\}. \tag{4}$$

The occurrence of the Möbius function in this formula permits the supposition that $a_{s,r}$ is very irregular. The initial values of $a_{s,3}$ for $s = 3, 4, \ldots, 20$ confirm this:

1, 3, 7, 12, 21, 31, 46, 62, 87, 109,

145, 178, 222, 266, 330, 381, 462, 530.

If we put $r = n+1$ in (4), we obtain a formula concerning the polynomials of the best approximation of degree n. Using the calculated numerical values of the function $a_{s,3}$, we see that for $n = 2$ the table of coefficients (1) with $s = 8, 12, 16, 20$ contains respectively 1, 3, 9, 18 pages, one page containing 360 coefficients. As we know, $s+1$ denotes the number of points of the set $\{0, 1, \ldots, s\}$ to which the points $t_0, t_1, \ldots, t_{n+1}$ belong.

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